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CONSTRUCTION OF RECTANGULAR PBIB DESIGNS

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Abstract

In this paper, some methods of constructing rectangular PBIB design are presented using the incidence matrix of a known PBIB design and Hadanard matrix. The rectangular design so constructed has nice block structure properties. The dual of these rectangular designs generate semi-regular GD designs. A rectangular design is also constructed by using $v = n^2$ treatments arranged in n$\times n$ square array. Dual of this design is again a rectangular design.

INTRODUCTION

A balanced incomplete block (BIB) design with parameters v, b, r, k, λ is a block design, BIBD (v, b, r, k, λ) with v treatments and b blocks of size k each such that every treatment occurs in exactly r blocks and that any two distinct treatments occur together in exactly λ blocks. This is a standard design used for constructing other designs (See Raghavarao, 1988).

A PBIB design, based on an s-associate association scheme, with parameters $v, b, r, k, \lambda_{i,} i = 1,2,3 \dots\dots..5,$ is a block design with v treatments and b blocks of size k each such that every treatment occurs in r blocks and any two distinct treatments being the i-th associate occur together in exactly λ_i blocks.

A group divisible (GD) design is a 2-associate PBIB design based on a group divisible association scheme, i.e. a set of $v = mn$ treatments can be divided into m groups of n treatments each such that any two treatments occur together in $\lambda_1$blocks if they belong to the same group, and in λ_2 blocks if they belong to different groups.

Rectangular designs, introduced vy Vartak (1995), are 3-associated PBIB designs based on rectangular association scheme of $v = mn$ treatments arranged in m$\times n$ rectangule array such that, with respect to each treatment, the first associates are the other n-1(=n_1 say) treatments of the same row, second associates are other m-1(=n_2 say) treatments of the same column and the remaining (m-1) (n-1) (=n_3 say) treatments are the third associates. That is a rectangular design in an arrangement of $v = mn$ treatments in b blocks such that (i) each block contain k distinct treatments k$< v,$ (ii) each treatment occurs in exactly r blocks, (iii) the mn treatments arranged in rectangle of m rows and n columns such that any two treatments in the same (column) occur together in $\lambda_1(\lambda_2)$ blocks respectively and inλ_3 blocks otherwise.

These designs have been studied by Bhagwandas et. al. (1985), Suen (1989), Sinha (1991), Kageyama and Miao (1995) and so on. The rectangular designs are useful as factorial experiments, having balance as well as orthogonality (Gupta and Mukerjee, 1989). In additions, if λ_3 is bigger thanλ_1 andλ_2 , loss of information on main effect becomes small, suen (1989), when these design are used as an m$\times$ n complete confounded factorial experiments.

In this paper, throughout, I_r denotes the identity matrix of order r,$J_{r\times s}$ denotes the r$\times$ s matrix all of whose elements are unity, $O_{r\times s}$ denotes the null matrix of order r$\times$ s, and A$\times$ B denotes Kronecker product of two matrix A and B.

CONSTRUCTIONS

It is known Vartak (1955) that the Kronecker Product of Incidence matrices of two BIB design produces a rectangular design. Here other variations will be considered.

The rectangular association scheme based on $v = mn$ treatments can be arranged in m row and n columns displayed in the following manner.

$$\begin{matrix} 1 & 2 & \ldots\ldots\ldots\ldots & n \\ n+1 & n+2 & \ldots\ldots\ldots\ldots & 2n \\ (m-1)\,n+1 & (m-1)n+2 & \ldots\ldots\ldots\ldots & mn \end{matrix}$$

Theorem 2.1: If N is the (0, 1) incidence matrix of order v $\times$ b of a self complementary BIB design with parameters v, b, r, k, λ then

$$M = \begin{pmatrix} O & N & \overline{N} & \overline{N} & N \\ N & O & N & \overline{N} & \overline{N} \\ \overline{N} & N & O & N & \overline{N} \\ \overline{N} & \overline{N} & N & O & N \\ N & \overline{N} & \overline{N} & N & O \end{pmatrix} \qquad \ldots (2.1)$$

yield a 3-associate PBIB design with parameters

$$v^* = 5v, \quad b^* = 5b, \quad r^* = 4r, \quad k^* = 4k, \quad \lambda_1^* = 4, \quad \lambda_2^* = r, \quad \lambda_3^* = 2r - \lambda$$

having a rectangular association scheme of

$n_1 = V - 1, \qquad n_2 = 4, \qquad n_3 = 4(V - 1),$

Proof: The positive parameters $v^*,\ b^*,\ \ r^*,\ \ k^*$ are obvious. Among $5v$ treatments a rectangular association scheme can be naturally defined as follows. These $5v$ treatments are arranged in a rectangular array of 5 rows and v columns such thst first associates of any treatment θ (say) are remaining $(v-1)$ treatments other than θ contained in the same row, the second associates are 4 treatments contained in the same column and the remaining 4 $(v-1)$ treatments are third associates of θ. The parameters λ_i^* (i=1, 2, 3) can be determined from the structure of MM'(M' be the transpose of M) in the following manner.

Suppose M be the (0, 1) incidence matrix of order $v^* \times b^*$ and has row (column) sum $r^*(k^*)$, then

$$MM' = \begin{pmatrix} A & B & B & B & B \\ B & A & B & B & B \\ B & B & A & B & B \\ B & B & B & A & B \\ B & B & B & B & A \end{pmatrix} \qquad \text{... (2.2)}$$

where

$$A = 2NN' + 2\bar{N}\bar{N}'$$
$$= 2NN' + 2(J - N)(J' - N')$$
$$= 2NN' + 2JJ' - 4NJ' + 2NN'$$
$$= 4NN' + 2\,bJ - 4rJ$$
$$= 4(r-\lambda)I + 4\lambda J, \qquad (\because b = 2r) \qquad \text{..... (2.3)}$$

and

$$\text{B} = \bar{N}N' + NN' + N\bar{N}'$$
$$= (J - N)N' + NN' + N(J' - N')$$
$$= 2NJ' - NN'$$
$$= 2rJ - (r-\lambda)I - \lambda J \qquad \text{...(2.4)}$$

From (2.2) and (2.4), we have

$$\lambda_1^* = 4\lambda, \qquad \lambda_3^* = (2r - \lambda)$$

In order to find the value of λ_2^* we consider the relation (2.4), we take the sum of coefficient of I and J in terms of r only. Hence we obtain

$$\lambda_2^* = 2r - r = r$$

Hence the theorem.

We study the combinatorial properties of the design constructed by Theorem 2.1.

Let us consider the characteristics roots of MM' given in (2.2) are given by $\theta_0 = r^*k^*, \theta_1, \theta_2$ and θ_3, where-

$$\theta_1 = r^* - \lambda_1^* + 4(\lambda_2^* - \lambda_3^*)$$
$$= 4r - 4\lambda + 4(r - 2r + \lambda)$$
$$= 0$$
$$\theta_2 = r^* - \lambda_2^* + (v-1)(\lambda_1^* - \lambda_3^*)$$
$$= 4r - r + (v-1)(4\lambda - 2r + \lambda)$$
$$= 3r + 5r(k-1) - 2vr + 2r$$

Since for BIBD, the relation r (k-1) =$\lambda(v-1)$ is true)

$$\theta_2 = 5r + 5rk - 5r - 4rk \qquad (\because v = 2k)$$
$$= rk$$
$$\theta_3 = r^* - \lambda_1^* - \lambda_2^* + \lambda^*$$
$$= 4r - 4\lambda - r + 2r - \lambda$$
$$= 5(r - \lambda)$$

It is well know that for a BIB design with $v = 2k$, r is always equal to $2\lambda + 1$, and hence

$$\theta_3 = \frac{5}{2}(r+1)$$

This number is always an integer if r is odd, further more, since r= $2\lambda + 1, so, r$ is always an odd integer.

Now it holds that in theorem 2.1

(i) $\lambda_1^* = \lambda_2^*$ iff $r = 4\lambda$
(ii) $\lambda_1^* = \lambda_3^*$ iff $r = \frac{5}{2}\lambda$
(iii) $\lambda_2^* = \lambda_3^*$ iff $r = \lambda$

Furthermore, it follows that

(a) Under (i) the rectangular PBIB design is reducible to an L_2 PBIB design if $v = 5$

(b) Under (ii) this design is reducible to aϵD design with v groups of 5 treatments each;

(c) Under (iii) the design is reducible to aϵD design with 5 groups of v treatments each.

Now, we discuss each of the above reducible cases.

(i) In case (a), we have r = 4λ, this relationis not possible in case of a BIB design with $v = 2k$ and Theorem 2.1 is not valid. Therefore the case (a) is impossible.

(ii) In case (b),we have r = $\frac{5}{2}\lambda\ i.e. 2r = 5\lambda$, we have only one BIBD (6,10,5,3,2) within the scope $v = 2k$, which does not yield any GD design. Hence this is also impossible.

(iii) In case (c) N is the incidence matrix of a complete blocks design for which $v = 2k$ is not true. This case is impossible.

Thus, these reduced cases of rectangular PBIB design in Theorem 2.1 are not interesting. Theorem 2.1 is thus useful as a real method of construction of rectangular PBIB designs.

Theorem 2.2: The existence of a normalized Hadamard matrix of order 4t (t$\geq$ 2) and a BIBD (v, b, r, k, λ) with $v = 2k$ implies the existence of a rectangular design with parameters

$$v^* = 4tv, \quad b^* = (4t-1)b, \quad r^* = (4t-1)r, \quad k^* = 4tk,$$

$$\lambda_1^* = (4t-1), \ \lambda_2^* = (2t-1)r, \quad \lambda_3^* = 2tr,$$

$$n_1 = v-1, \ n_2 = 4t-1, \quad n_3 = (4t-1)(v-1) \quad \ldots (2.5)$$

by detecting first column of normalized Hadamard matrix Hut and replacing 1 by N and -1 by N in the remaining matrix of order 4t $\times$ $(4t-1)$.

Proof: The proof of the theorem is obvious.

As an illustration, consider the BIB designs(6,10,5,3,2) and (8,14,7,4,3), we obtain following rectangular designs with parameters

(i) $v^* = 48, b^* = 70, \ r^* = 35, \ k^* = 24, \ \lambda_1^* = 14, \lambda_2^* = 15, \lambda_3^* = 18, n_1 = 5, n_2 = 7, n_3 = 35$

(ii) $v^* = 64, b^* = 98,\ r^* = 49,\ k^* = 32,\ \lambda_1^* = 21, \lambda_2^* = 21, \lambda_3^* = 25, n_1 = n_2 = 7, n_3 = 49$

Below we list the semi-regular GD designs which are obtained on taking the dual of rectangular PBIB designs with parameters.

The dual of certain PBIB design with three associate classes with $v > b$ and values 0 and 1 for $\lambda_1, \lambda_2, \lambda_3$ have also been obtained by Shrikhande and Bhagwandas (1965). They have show that dual of PBIB design with three associate classes is PBIB design with two associate classes with $\lambda_1 = 1, \lambda_2 = 0\ and\ P_{11}^2 = 0$

Theorem 2.3: The existence of MOLS of order 5 implies the existence of rectangular PBIB design with parameters $= b = s^2, r = k = s,\ \lambda_1 = 1,\ \lambda_{2,} = 0, \lambda_3 = 1, m = n = s.$

Proof: Consider the rectangular association scheme for $v = s^2$ treatments in a square array of order s$\times s$ array. Then consider all the latin square $L_1, L_2, L_3 \dots\dots\dots\dots\dots\dots\dots\dots L_{s-1}$ of order S. Superimpose the square array s$\times s$ on the latin squares of order s. Write the blocks taking treatments occurring with the same letter of latin squares, we get in all s+s(s-1)=s^2 blocks of the design the values of r and k are obvious treatments which are contained in the same row of the rectangular association scheme are first associate, they are occurring exactly λ_1 times in the design. In similar manner, we can obtain the value of $\lambda_{2,} = 0\ and\ \lambda_3 = 1$. Hence the theorem.

Theorem 2.4: If N is the incidence matrix of a BIBD (v, b, r, k, λ), then the incidence pattern

$$S = \begin{pmatrix} O & N & N & N \\ N & O & N & N \\ N & N & O & N \\ N & N & N & O \end{pmatrix} \qquad \dots\dots\dots\dots (2.7)$$

is the incidence matrix of a rectangular PBIB design with three associate classes having the parameters

$$v^* = 4v,\ b^* = 4b, r^* = 3r, k^* = 3k, \lambda_1^* = 3\lambda,\ \lambda_2^* = 2r, \lambda_3^* = 2\lambda,$$

$$n_1 = v - 1, n_2 = 3, n_3 = 3(v - 1) \qquad \dots.. (2.8)$$

Proof: The values of the parameters $v^*,\ b^*, r^*, k^*$ are obvious. Among $4v$ treatments the rectangular association scheme can be simply discussed as follows. The $4v$ treatments are arranged in an array of 4 rows and v columns in such a way that first associate of any particular treatment θ(say) are others $(v-1)$ treatments contained in the same row, the second associates are those 3treatments which are contained in the same column and the remaining 3 $(v-1)$ treatments are third associates of θ.

The values of λ_i (i=1, 2, 3) can be easily calculated from the structure of SS' in the following way.

Let S be the (0,1) incidence matrix of order $v^* \times b^*$ and has column (row) sum $k^*(r^*)$ and S' be the transpose of S, then

$$SS' = \begin{pmatrix} C & D & D & D \\ D & C & D & D \\ D & D & C & D \\ D & D & D & C \end{pmatrix} \quad \ldots (2.9)$$

where

$$C = 3NN'$$
$$= 3(r-\lambda)I + 3\lambda J, \quad \ldots. (2.10)$$

and

$$D = 2NN'$$
$$= 2(r-\lambda)I + 2\lambda J \quad \ldots (2.11)$$

From (2.10) and (2.11), we get,

$$\lambda_1^* = 3\lambda,\ \lambda_2^* = 2r, \quad \lambda_3^* = 2\lambda,$$

The characteristic roots of SS' are

$$\theta_0 = rk$$
$$\theta_1 = r - \lambda_1 + (m-1)(\lambda_2 - \lambda_3)$$
$$\theta_2 = r - \lambda_2 + (n-1)(\lambda_2 - \lambda_3)$$
$$\theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3.$$

For further combinational properties of rectangular design we refers to Shah (1964) and Vartak (1955, 1959)

$$\theta_0 = r^* k^*$$

$$\theta_1 = 9(r - \lambda)$$

$$\theta_2 = r(2v - 2k + 1)$$

$$\theta_3 = (r - \lambda)$$

REFERENCES

1. Bhagwandas, Banerjee, S. and Kageyama, S. (1985): Patterened constructions of partially balanced incomplete block designs. Comm. Statist. Theory Meth. A, 14, 1259-1267.

2. Gupta, S. and Mukerjee, R. (1989): A calculus for factorial arrangements, Lecture Notes in Statistics, Vol. 59, Springer- Verlag, New York.

3. Kageyama and Miao, Y. (1995): Some methods of constructions of rectangular PBIB designs, combinatories Theory, 1, 186-198.

4. Raghavaro, D. (1988): Constructions and Combinatiorial problems in Design of Experiments, Dover, New York.

5. Shah, S.M. (1964): An upper bound for the number of disjoint blocks in certain PBIB designs. Ann. Math. Stat. 35, 398-407.

6. Shrikhande, S.S. and Bhagwandas (1965): Duals of incomplete block designs. J. Indian Stat. Assn. 3, 30-37.

7. Suen, C. (1989): Some rectangular designs constructed by the method of differences, j. Statist. Plam. Inference, 27, 273-275.

8. Sinha, K. (1991): A construction of rectangular designs, J. Comb. Math. Comb. Computing, 9, 199-200.

9. Vartak, M.N. (1955): On an application of Kronecker product of Matrices to Statistical designs. Ann. Math. Stat. 26, 420-438.

10. Vartak M.N. (1959): The non-existence of certain PBIB designs Ann. Math. Stat, 30, 1051-1062.