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COMPARISON BETWEEN BAYESIAN AND MAXIMUM LIKELIHOOD ESTIMATION OF SCALE PARAMETER IN WEIBULL DISTRIBUTION WITH KNOWN SHAPE UNDER LINEX LOSS FUNCTION

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Abstract

Weibull distribution is widely employed in modeling and analyzing lifetime data. The present paper considers the estimation of the scale parameter of two parameter Weibull distribution with known shape. Maximum likelihood estimation is discussed. Bayes estimator is obtained using Jeffreys' prior under linex loss function. Relative efficiency of the estimators are calculated in small and large samples for over-estimation and under-estimation using simulated data sets. It is observed that Bayes estimator fairs better especially in small sample size and when over estimation is more critical than under estimation.

INTRODUCTION

The Weibull distribution is one of the most widely used distributions for analyzing lifetime data. It is found to be useful in diverse fields ranging from engineering to medical sciences (see Lawless [4], Martz and Waller [6]). The Weibull family is a generalization of the exponential family and can model data exhibiting monotone hazard rate behavior, i.e. it can accommodate three types of failure rates, namely increasing, decreasing and constant. The probability density function of the Weibull distribution is given by:

$$f(x|\alpha,\beta) = \frac{\beta}{\alpha} x^{\beta-1} \exp[-\frac{x^{\beta}}{\alpha}] \; ; \quad x \geq 0,\ \alpha,\ \beta > 0 \qquad (1)$$

where the parameter β determines the shape of the distribution and α is the scale parameter. In Weibull lifetime analysis it is frequent case that the value of the shape parameter is known. For example, the exponential (α) and Rayleigh distributions are obtained when β=1 and β=2, respectively. Soland [9] gives a justification for this situation.

The maximum likelihood (ML) method of estimation is quite efficient and very popular. In Bayesian approach, a prior distribution for the parameter is considered and then the posterior distribution is obtained by conditioning on the data and after that the inference is done based on the posterior. Ahmed *et al*.[1], have considered ML and Bayesian estimation of the scale parameter of Weibull distribution with known shape and compared their performance under squared error loss.

The squared error loss denotes the punishment in using $\hat{\theta}$ to estimate θ and is given by L $(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$.This loss function is symmetric in nature i.e. it gives equal weightage to both over and under estimation. In real life, we encounter many situations where over-estimation may be more serious than under-estimation or vice versa. Varian [10] introduced a very useful asymmetric linex loss function given by

$$L(\delta) = \exp[a\delta]-a\delta-1; \quad \delta = \hat{\theta}-\theta, \; a \neq 0 \tag{2}$$

Here, a determines the shape of the loss function. For a>0, over-estimation is more heavily penalized, the same being true for under-estimation when a<0. For $|a| \to 0$, this loss is almost symmetric and not far from a squared error loss function. Zellner [11] discussed Bayesian estimation and prediction using linex loss. The invariant form of the linex loss function (see Pandey [7], Pandey *et al*.[8]) which is more suitable in the case of estimation of scale parameter is given by

$$L(\delta) = \exp[a\delta]-a\delta-1; \quad \delta = \frac{\hat{\theta}}{\theta}-1, \; a \neq 0 \tag{3}$$

In estimating mean time to failure, over-estimation and under-estimation should not be given equal importance. Over-estimation could lead to fixing a guarantee period beyond the true mean lifetime, which in turn could lead to considerable loss for the producer. Thus, an asymmetric loss, giving unequal weightage to over and under estimation, seems more appropriate for this problem.

In this paper, ML estimator and Bayes estimator of the scale parameter of the Weibull distribution is considered under loss (3), with the assumption that the shape parameter is known. The plan of the paper is as follows. In section 2, the ML estimation of α is reviewed. Section 3 is devoted to the derivation of the Bayes estimator under invariant form of the linex loss using Jeffreys' prior. In section 4, a simulation study is discussed and results are presented. Concluding remarks are presented in section 5.

1. Maximum Likelihood Estimation

Maximum likelihood estimation of the parameters of Weibull distribution is well discussed in literature (see Cohen [2] and Mann *et al*.[5]).

Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a sample of size n from a Weibull distribution with parameters α and β. The likelihood function is given by

$$L(\alpha, \beta \mid \underline{x}) = \frac{\beta^n}{\alpha^n} \prod_{i=1}^{n} x_i^{\beta-1} \exp\left[-\frac{x_i^{\beta}}{\alpha}\right]$$

As the shape parameter β is assumed to be known, the ML estimator of α is obtained by solving the equation

$$\frac{\partial \log L(\alpha, \beta \mid \underline{x})}{\partial \alpha} = 0,$$

which gives us

$$-\frac{n}{\alpha} + \frac{\sum_{i=1}^{n} x_i^{\beta}}{\alpha^2} = 0 \quad \Rightarrow \alpha = \frac{\sum_{i=1}^{n} x_i^{\beta}}{n}$$

Thus the ML estimator of α is given by

$$\hat{\alpha}_m = \frac{\sum_{i=1}^{n} x_i^{\beta}}{n} \tag{4}$$

2. Bayesian Estimation

THE PRIOR DISTRIBUTION

Quite often, the derivation of the prior distribution based on information other than the current data is impossible or rather difficult. Moreover, the statistician may be required to employ as little subjective input as possible, so that the conclusion may appear solely based on sampling model and the current data.

Jeffreys [3] proposed a formal rule for obtaining a non-informative prior as

$$g(\theta) \propto \sqrt{|\det I(\theta)|}$$

where θ is k-vector valued parameter and $I(\theta)$ is the Fisher's information matrix of order k $\times$k. In particular, if θ is a scalar parameter, Jeffreys' non-informative prior for θ is $g(\theta) \propto \sqrt{I(\theta)}$. Thus, in our problem, we consider the prior distribution of α to be

$$g(\alpha) \propto \sqrt{I(\alpha)} \Rightarrow g(\alpha) = k.\frac{1}{\alpha}.$$

where k is a constant.

The posterior distribution of α is given by

$$\pi(\alpha \mid \underline{x}) = \frac{f(\underline{x} \mid \alpha) g(\alpha)}{\int_0^\infty f(\underline{x} \mid \alpha) g(\alpha) d\alpha}$$

where $f(\underline{x} \mid \alpha)$ is the joint density of $\underline{x}$ and is given by

$$f(\underline{x} \mid \alpha) = \left(\frac{\beta}{\alpha}\right)^n \left(\prod_{i=1}^n x_i^{\beta-1}\right) \exp\left[-\frac{\sum_{i=1}^n x_i^\beta}{\alpha}\right]$$

Using the transformation $\frac{\sum_{i=1}^n x_i^\beta}{\alpha} = z$, a straightforward integration gives us

$$\int_0^\infty f(\underline{x} \mid \alpha) g(\alpha) d\alpha = k \beta^n \Gamma n \frac{\left(\prod_{i=1}^n x_i\right)^{\beta-1}}{\left(\sum_{i=1}^n x_i^\beta\right)^n}$$

Hence, the posterior distribution of α is given by

$$\pi(\alpha \mid \underline{x}) = \frac{\exp\left[-\frac{\sum_{i=1}^{n} x_i^{\beta}}{\alpha}\right]\left(\sum_{i=1}^{n} x_i^{\beta}\right)^n}{\alpha^{(n+1)}\Gamma n} \tag{5}$$

ESTIMATION UNDER LINEX LOSS

To obtain the Bayes estimator, we minimize the posterior expected loss given by

$$\rho = \int_0^{\infty} (\exp[a\delta] - a\delta - 1)\frac{\exp[-\frac{t}{\alpha}]t^n}{\alpha^{(n+1)}\Gamma n}d\alpha$$

where $\delta = \frac{\hat{\alpha}}{\alpha} - 1$ and t = $\sum_{i=1}^{n} x_i^{\beta}$, with respect to $\hat{\alpha}$. Integrating, we have

$$\rho = \frac{t^n \exp[-a]}{(t - a\hat{\alpha})^n} - \frac{na\hat{\alpha}}{t} + (a-1)\,.$$

Solving $\frac{\partial \rho}{\partial \hat{\alpha}} = 0$, we obtain the Bayes estimator as

$$\hat{\alpha}_b = \frac{t}{a}\left(1 - \exp\left[-\frac{a}{n+1}\right]\right) \tag{6}$$

3. Simulation Study

In this study, we have generated random samples from Weibull distribution and compared the performance of ML and Bayes estimator based on them. We have chosen sample size n = 5, 10, 15, 20, 50, 100 to represent both small and large sample, unlike Ahmed *et al.*[1], who have considered only large sample size. For the scale parameter, we have considered α = 0.5 and 1.5. The shape β has been fixed at 0.8, 1.0 and 1.2, representing decreasing, constant and increasing hazard rates respectively. All six possible combinations of the parameters have been considered. We have taken a=1,2,-1,-2 in linex loss considering varying weightage for over-estimation and under-estimation. The number of replications used was M=1000. The risk of the estimators were calculated by the formula

$$\text{Risk}(\hat{\alpha}) = \frac{\sum_{i=1}^{M}\left[\exp\left[a\left(\frac{\hat{\alpha}_i}{\alpha}-1\right)\right]-a\left(\frac{\hat{\alpha}_i}{\alpha}-1\right)-1\right]}{M}$$

The relative efficiency of the Bayes estimator with respect to the ML estimator is given by

$$\text{RE}=\frac{Risk(\hat{\alpha}_m)}{Risk(\hat{\alpha}_b)} \tag{7}$$

The simulation was run using the package R v.(2.9.1) (freely available from *http://www.r-project.org*)

Table 1-6, appended at the end of the paper, give the relative efficiency of the Bayes estimator with respect to the ML estimator. Each table corresponds to a particular combination of parameter values and gives the relative efficiency for different sample sizes over 4 different values of a.

We note that Bayes estimator fairs better than or equal to ML estimator in general, whereas under squared error loss function Ahmed *et al.*[1] concluded that ML estimator is better than Bayes estimator. In particular, the Bayes estimator outperforms the ML estimator in small sample size when over-estimation gets more weightage than under-estimation. In large samples, the estimators are almost equally efficient.

4. Concluding Remarks

The present paper explores ML and Bayesian estimation of scale parameter in Weibull distribution under linex loss and demonstrates that the Bayes estimator performs better than the ML estimator when over-estimation is of more importance and when sample size is small. In life testing, situations may arise where over-estimation is more critical than under-estimation and should receive very high weightage. For example, fixing the guarantee period for products, as discussed in section 1, could get much more importance when the item is very costly, like an engine for an aeroplane or a picture tube of a television set, as even a single replacement would amount to a considerable loss for the manufacturer. Also, in these cases we are more likely to work with small samples, because of the cost restricting the number of items put to life test. Based on the present study, the use of Bayes estimator in these scenarios is recommended.

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Table 1: RE under linex loss for α =0.5, β =0.8

n	a=1	a=2	a= -1	a= -2
5	2.414496	4.119995	1.184758	1.014186
10	1.766213	2.468274	1.159609	1.006613
15	1.639807	2.015275	1.130456	1.004604
20	1.520482	1.806507	1.119393	1.003126
50	1.2883	1.436478	1.07956	1.000973
100	1.178693	1.255588	1.052198	1.000326

Table 2: RE under linex loss for α =0.5, β =1.0

n	a=1	a=2	a= -1	a= -2
5	1.639668	2.09538	1.049588	1.001127
10	1.283212	1.604791	1.035481	1.000171
15	1.196475	1.306765	1.023396	1.000398
20	1.166145	1.247874	1.002313	0.999761
50	1.04104	1.090504	1.001522	0.999990
100	1.024122	1.039281	1.000993	1.000036

Table 3: RE under linex loss for α =0.5, β =1.2

n	a=1	a=2	a= -1	a= -2
5	1.106725	1.280404	0.926309	0.987624
10	0.899054	0.912025	0.916089	0.994025
15	0.859699	0.836261	0.917235	0.995848
20	0.812855	0.806697	0.921755	0.997075
50	0.832358	0.805112	0.936302	0.999111
100	0.883368	0.857083	0.955831	0.9997

Table 4: RE under linex loss for α =1.5, β =0.8

n	a=1	a=2	a= -1	a= -2
5	1.129454	1.387942	0.970119	0.994430
10	0.992871	1.085951	0.939167	0.996460
15	0.920736	0.923472	0.942921	0.997167
20	0.903386	0.886892	0.943230	0.997957
50	0.874424	0.832997	0.943329	0.999283
100	0.857839	0.853951	0.956492	0.999693

Table 5: RE under linex loss for α =1.5, β =1.0

n	a=1	a=2	a= -1	a= -2
5	1.521064	2.173724	1.045962	0.999895
10	1.354525	1.530984	1.029179	0.999793
15	1.222593	1.289258	1.031999	1.000056
20	1.141672	1.257265	1.015046	1.000141
50	1.062012	1.095119	1.003041	0.999934
100	1.015819	1.034365	0.999954	0.999988

Table 6: RE under linex loss for α =1.5, β =1.2

n	a=1	a=2	a= -1	a= -2
5	1.981569	3.29179	1.124522	1.007063
10	1.652204	2.215139	1.106804	1.004386
15	1.434133	1.715492	1.096144	1.003605
20	1.387536	1.616163	1.089482	1.001954
50	1.260439	1.365434	1.073076	1.000823
100	1.179093	1.270592	1.055923	1.000332