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ON A GENERATING FUNCTION FOR THE HERMITE POLYNOMIALS

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Abstract

Talman [1] used a Group theoretic approach to obtain an interesting expression between Hermite and Laguerre polynomials. Here we show that the Talman´s relation permits to deduce easily the generating function of Saha [2] for Hermite polynomials.

INTRODUCTION

Talman [1] employed Group theory to find an expression between Laguerre and Hermite polynomials, in fact:

$$\sum_{\rho=0}^{\infty} \frac{\left(te^{i\alpha}\right)^{\rho}}{\rho!} L_{\mu}^{\rho-\mu}\left(2t^2\right) H_{\rho}(\xi) = \frac{\left(te^{i\alpha}\right)^{\mu}}{\mu!} H_{\mu}(\xi - 2t\cos\alpha) \cdot \exp\left(2\xi te^{i\alpha} - t^2 e^{2i\alpha}\right) \qquad (1)$$

where our notation is the same than Abramowitz-Stegun [3]. In the next section we shall show that under simple manipulations the relation (1) implies the following generating function for Hermite polynomials:

$$\exp\left(2x\eta - \eta^2\right) = \sum_{\mu=0}^{\infty} \frac{\left[\left(\gamma - \sqrt{\gamma^2 - 1}\right)\eta\right]^{\mu}}{\mu!} H_{\mu}\left[\left(\gamma + \sqrt{\gamma^2 - 1}\right)x - \sqrt{\gamma^2 - 1} \cdot \eta\right] \qquad (2)$$

that Saha [2] obtains by another route without Group theory.

The study of formulae involving Laguerre and Hermite polynomials has great importance in the analysis of several quantum mechanical problems [4-8].

Saha´s generating function

From definition of the associated Laguerre polynomials [3] it is immediate the property:

$$\sum_{\mu=0}^{\infty} L_{\mu}^{\rho-\mu}(y) = 2^{\rho} e^{-y} , \tag{3}$$

then we apply to (1) the operation $\sum_{\mu=0}^{\infty}$ and we use (3) to obtain:

$$\sum_{\mu=0}^{\infty} \frac{\left(te^{i\alpha}\right)^{\mu}}{\mu!} H_{\mu}(\xi - 2t\cos\alpha) = \exp\left(2\xi te^{i\alpha} - 3t^2 e^{2i\alpha} - 2t^2\right) \tag{4}$$

where we have employed the well known generating function [3]:

$$e^{2yz-z^2} = \sum_{\rho=0}^{\infty} \frac{z^{\rho}}{\rho!} H_{\rho}(y) \tag{5}$$

If in (4) we make the changes $t = i\eta$, $\alpha = -\varphi$ and $\xi = -ixe^{i\varphi} + 3i\eta\cos\varphi$ with η , φ , x reals, then it results the relation:

$$e^{2x\eta-\eta^2} = \sum_{\mu=0}^{\infty} \frac{\left(i\eta e^{-i\varphi}\right)^{\mu}}{\mu!} H_{\mu}\left(-ixe^{i\varphi} + i\eta\cos\varphi\right) \tag{6}$$

which contains (5) when $\varphi = \frac{\pi}{2}$; the complex conjugate of (6) is given by:

$$\exp\left(2x\eta - \eta^2\right) = \sum_{\mu=0}^{\infty} \frac{\left(-i\eta e^{i\varphi}\right)^{\mu}}{\mu!} H_{\mu}\left(ixe^{-i\varphi} - i\eta\cos\varphi\right) \tag{7}$$

and, finally, if we introduce the notation $\gamma = \sin\varphi$ then (7) leads to the generating function (2) deduced by Saha, q. e. d.

REFERENCES

1. J.D. Talman, Special functions: A Group theoretic approach, W.A. Benjamin Inc., N.Y. (1968) Chap. 13.

2. B.B. Saha, On a generating function of Hermite polynomials, Yokohama Math. J. **27** (1969) 73-76

3. M. Abramowitz and I.A. Stegun, Handbook of mathematical functions, Wiley and Sons, N.Y. (1972) Chap. 22.

4. J. López-Bonilla, D. Navarrete, H. Núñez-Yépez and A. Salas-Brito, Oscillators in one and two dimensions and ladder operators for the Morse and the Coulomb problems, Int. J. Quantum Chem. **62,** No.2 (1997) 177-183

5. G.F. Torres del Castillo and A. López-Villanueva, Interbasis expansion and SO(3) symmetry in the two-dimensional hydrogen atom, Rev. Mex. Fís. **47,** No.2 (2001) 123-127

6. J. López-Bonilla, A. Lucas-Bravo and S. Vidal, Integral relationship between Hermite and Laguerre polynomials: Its application in quantum mechanics, Proc. Pakistan Acad. Sci. **42**, No.1 (2005) 63-65

7. V. Gaftoi, J. López-Bonilla and G. Ovando, Matrix elements for the one-dimensional harmonic oscillator and Morse's radial wave equation, South East Asian J. Math. & Math. Sci. **4**, No.1 (2005) 61-64

8. L. Cruz-Beltrán and J. López-Bonilla, Two-dimensional harmonic oscillator: Its ladder operators, The Icfai Univ. J. Phys. **1**, No.3 (2008) 17-19.