TRANSMISSION OF INFECTIOUS DISEASES BY DROPLET INFECTION AND THEIR CONTROL BY AGE SPECIFIC IMMUNIZATION: A DELAY MODEL

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Abstract

Infectious diseases of childhood that spread mostly by droplet infection are Chickenpox, Measles, Diphtheria and Whooping cough. In order to study the spread of these respiratory diseases, a delay mathematical model has been proposed and analyzed using stability theory. In the proposed model the underlying population has been divided into two subpopulations consisting of infants and juveniles. For the control of the disease it has been assumed in the model that only infants are vaccinated at a constant rate. Since in the target population, age distribution is considered, a delay in maturation rate has been incorporated in the model. The model has been analyzed by conducting the linear and non-linear stability analysis of the disease free and endemic equilibrium points. On the basis of the asymptotic long term analysis, criteria for the spread and control of the disease have been derived.

Keywords

Maturation delay, Disease frees equilibrium, Endemic equilibrium, Stability, Vaccination, Droplet infection.

Introduction

Infectious diseases of childhood that spread mostly by droplet infection are Chickenpox, Measles, Diphtheria and Whooping cough. Droplet infection is direct projection of a spray of droplets of saliva and naso-pharyngel secretions during coughing, sneezing or speaking and spitting in to surrounding atmosphere. The expelled droplets may impinge directly upon the conjunctiva, oro-respiratory mucosa or skin of a close contact.

In the above-mentioned infectious diseases these droplets which contain millions of bacteria and viruses can be the source of infection to others. When a healthy susceptible person comes within the range of these infected droplets, he is likely to inhale some of them and acquire infection.

All these above-mentioned diseases occur primarily among children under the age of 10 years and one attack generally confers life-long immunity. In all the four diseases referred above immunity after vaccination is long-lasting. In order to study the dynamics of these kinds of disease, age dependent epidemic mathematical models have to be constructed. Some age-dependent epidemic models have been studied mainly by Hethcote [4], Anderson and May [1] and Bussenberg and Castillo-Chavez [2]. Tchuenche et al [7] have studied global behaviour of an SIR epidemiological model with time delay. Jin Z and Ma Z [5] have studied the stability of an SIR epidemic model

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with time delay. Shujing et al [3] have studied impulsive vaccination of an SEIRS model with time delay. In these models age specific vaccination and maturation delay have not been considered. Only recently Misra et. al [6] have studied effects of age based vaccination on the dynamics of delay epidemic model.

In view of the above a delay mathematical model has been considered in this paper to study the transmission of infectious diseases by droplet infection and their control by age specific immunization.

In the formulation of the proposed mathematical model the underlying population has been divided in to two age groups consisting of infants and juveniles because only the populations of these two age groups are being affected by the disease under consideration. For controlling of the disease it has been assumed that only infants are vaccinated at a constant rate as this is being observed in some of the vaccination policies. Since in the target population age distribution is considered, a delay in maturation rate has been also incorporated to make the model more realistic. With these assumptions the mathematical model has been constructed which is being given by the following system of non-linear ordinary differential equations.

Mathematical model 1

$$\frac{dS_1}{dt} = \Lambda - \beta_1 S_1 P - (d+\mu)S_1 - mS_1(t-T) \tag{1}$$

$$\frac{dI_1}{dt} = \beta_1 S_1 P - (d+\gamma_1)I_1 - mI_1(t-T) \tag{2}$$

$$\frac{dR_1}{dt} = \gamma_1 I_1 + \mu S_1 - dR_1 - mR_1(t-T) \tag{3}$$

$$\frac{dS_2}{dt} = -\beta_2 S_2 P + mS_1(t-T) - dS_2 \tag{4}$$

$$\frac{dI_2}{dt} = \beta_2 S_2 P - (d+\gamma_2)I_2 + mI_1(t-T) \tag{5}$$

$$\frac{dR_2}{dt} = \gamma_2 I_2 - dR_2 + mR_1(t-T) \tag{6}$$

$$\frac{dP}{dt} = w(I_1 + I_2) - \delta P \tag{7}$$

with the initial conditions

$$\begin{aligned} &S_1(0) = S_{10} > 0 \qquad && I_1(0) = I_{10} > 0 \qquad && R_1(0) = 0 \\ &S_2(0) = S_{20} > 0 && I_2(0) = I_{20} > 0 && R_2(0) = 0 \\ &P(0) = P_0 > 0 && && \end{aligned} \tag{8}$$

where,

S_1 = Susceptible class consisting of infants

S_2 = Susceptible class consisting of juveniles

I_1 = Infective class consisting of infants

I_2 = Infective class consisting of juveniles

R_1 = Removed class consisting of infants

R_2 = Removed class consisting of juveniles

P = Infective pathogen or infectious agents

□ = Recruitment rate.

d = Natural death rate of humans.

□ $_i$ = Transmission rates of infection for i = 1,2

□ $_i$ = Removal rates for i = 1,2

□ = Death rate of pathogen.

□ = Vaccination rate

w = The rate at which infective individual produces pathogen.

m = Maturation rate

T = Maturation delay

Equilibrium points:

The two equilibrium points of the model are:

(i) Disease free equilibrium point

$E_0(\overline{S}_1, \overline{I}_1, \overline{R}_1, \overline{S}_2, \overline{I}_2, \overline{R}_2, \overline{P})$, where

$$\overline{S}_1 = \Lambda/(d+\mu+m), \overline{I}_1 = 0, \ \overline{R}_1 = \mu\Lambda/(d+m)(d+m+\mu)$$

$$\overline{S}_2 = m\Lambda/d(d+\mu+m), \ \overline{I}_2 = 0, \ \overline{R}_2 = m\mu\Lambda/d(d+m)(d+m+\mu) \ \& \ \overline{P} = 0$$

(ii) Endemic equilibrium point

$E_1(S_1^*, I_1^*, R_1^*, S_1^*, I_1^*, R_1^*, P^*)$, where

$$S_1^* = \Lambda/(d+m+\mu+\beta_1 P^*), \ I_1^* = \beta_1 S_1^* P^*/(\gamma_1+d+m),$$

$$R_1^* = (\mu S_1^* + \gamma_1 I_1^*)/(d+m), \ S_2^* = mS_1^*/(\beta_2 P^* + d),$$

$$I_2^* = (\beta_2 S_2^* P^* + mI_1^*)/(\gamma_2+d), \ \ R_2^* = (mR_1^* + \gamma_2 I_2^*)/d \ \&$$

$$P^* = \left\{-q \pm \sqrt{q^2 - 4pr}\right\}/2p \text{, provided } q^2 > 4pr$$

where ,

$$p = \delta(\gamma_2 + d)(\gamma_1 + d + m)\beta_1\beta_2,$$

$$q = \delta(\gamma_2 + d)(\gamma_1 + d + m)\{(d + m + \mu)\beta_2 + d\beta_1\} - w\Lambda(\gamma_2 + d + m)\beta_1\beta_2$$

$$r = \delta(\gamma_2 + d)(\gamma_1 + d + m)(d + m + \mu)d - w\Lambda\{(\gamma_2 + d + m)\beta_1 d + (\gamma_1 + d + m)\}\beta_2 m$$

Before analyzing the main model we will present a brief discussion of the following mathematical models (a) and (b) which are special cases of the main model. In the model (a) vaccination and delay has not been considered and in model (b) vaccination has been considered at a constant rates but delay has not been taken into account.

Sub Model (a):

$$\frac{dS_1}{dt} = \Lambda - \beta_1 S_1 P - (d + m)S_1$$

$$\frac{dI_1}{dt} = \beta_1 S_1 P - (d + \gamma_1 + m)I_1$$

$$\frac{dR_1}{dt} = \gamma_1 I_1 - (d + m)R_1$$

$$\frac{dS_2}{dt} = -\beta_2 S_2 P + mS_1 - dS_2$$

$$\frac{dI_2}{dt} = \beta_2 S_2 P - (d + \gamma_2)I_2 + mI_1$$

$$\frac{dR_2}{dt} = \gamma_2 I_2 - dR_2 + mR_1$$

$$\frac{dP}{dt} = w(I_1 + I_2) - \delta P$$

With respect to the above model (a) we put the following results:

(i) The disease free equilibrium point is linearly stable if following conditions are satisfied.

$$\begin{aligned}
&2d + m - \beta_1\bar{S}_1 > 0 \\
&2d + \gamma_1 + m - \beta_1\bar{S}_1 - w > 0 \\
&2d - \gamma_1 + m > 0 \\
&2d - \beta_2\bar{S}_2 - m > 0 \\
&2d + \gamma_2 - \beta_2\bar{S}_2 - m - w > 0 \\
&2d - \gamma_2 - m > 0 \\
&\delta - w - \beta_1\bar{S}_1 - \beta_2\bar{S}_2 > 0
\end{aligned} \tag{9}$$

(ii) The endemic equilibrium point is linearly stable if following conditions are satisfied.

$$
\begin{aligned}
&2d + m + \beta_1 P^* - \beta_1 S_1^* > 0 \\
&2d + \gamma_1 + m - \beta_1 P^* - \beta_1 S_1^* - w > 0 \\
&2d - \gamma_1 + m > 0 \\
&2d - \beta_2 P^* - \beta_2 S_2^* - m > 0 \\
&2d + \gamma_2 - \beta_2 P^* - \beta_2 S_2^* - m - w > 0 \\
&2d - \gamma_2 - m > 0 \\
&\delta - w - \beta_1 S_1^* - \beta_2 S_2^* > 0
\end{aligned}
\tag{10}
$$

(iii) The disease free equilibrium point is non- linearly stable in the region given by

$$
\begin{aligned}
&\mathrm{D} = \{(\mathrm{S}_1, \mathrm{I}_1, \mathrm{R}_1, \mathrm{S}_2, \mathrm{I}_2, \mathrm{R}_2, \mathrm{P}) : 0 < \mathrm{S}_1 + \mathrm{I}_1 + \mathrm{R}_1 \le \alpha_1, 0 < \mathrm{S}_2 + \mathrm{I}_2 + \mathrm{R}_2 \le \alpha_2 \ \& \ \mathrm{P} > 0 \\
&\text{where, } \alpha_1 = \Lambda/(\mathrm{d}+\mathrm{m}) \ \& \ \alpha_2 = \mathrm{m}\Lambda/\mathrm{d}(\mathrm{d}+\mathrm{m})\}
\end{aligned}
$$

if following conditions are satisfied

$$
\begin{aligned}
&2d + m - \beta_1 \Lambda/(d+m) > 0 \\
&2d + \gamma_1 + m - \beta_1 \Lambda/(d+m) - w > 0 \\
&2d - \gamma_1 + m > 0 \\
&2d - \beta_2 m\Lambda/d(d+m) - m > 0 \\
&2d + \gamma_2 - \beta_2 m\Lambda/d(d+m) - m - w > 0 \\
&2d - \gamma_2 - m > 0 \\
&\delta - w - (\beta_2 m/d + \beta_1)\Lambda/(d+m) > 0
\end{aligned}
\tag{11}
$$

(iv) The endemic equilibrium point is non-linearly stable in the region D if following conditions are satisfied.

$$
\begin{aligned}
&2d + m + \beta_1 P^* - \beta_1 \Lambda/(d+m) > 0 \\
&2d + \gamma_1 + m - \beta_1 P^* - \beta_1 \Lambda/(d+m) - w > 0 \\
&2d - \gamma_1 + m > 0 \\
&2d + \beta_2 P^* - \beta_2 m\Lambda/d(d+m) - m > 0 \\
&2d + \gamma_2 - \beta_2 P^* - \beta_2 m\Lambda/d(d+m) - m - w > 0 \\
&2d - \gamma_2 - m > 0 \\
&\delta - w - (\beta_2 m/d + \beta_1)\Lambda/(d+m) > 0
\end{aligned}
\tag{12}
$$

Sub Model (b):

$$
\frac{dS_1}{dt} = \Lambda - \beta_1 S_1 P - (d + m + \mu)S_1
$$

$$
\frac{dI_1}{dt} = \beta_1 S_1 P - (d + \gamma_1 + m)I_1
$$

$$\frac{dR_1}{dt} = \gamma_1 I_1 + \mu S_1 - (d+m)R_1$$
$$\frac{dS_2}{dt} = -\beta_2 S_2 P + mS_1 - dS_2$$
$$\frac{dI_2}{dt} = \beta_2 S_2 P - (d+\gamma_2)I_2 + mI_1$$
$$\frac{dR_2}{dt} = \gamma_2 I_2 - dR_2 + mR_1$$
$$\frac{dP}{dt} = w(I_1 + I_2) - \delta P$$

With respect to the above model (b) we put the following results:

(i) The disease free equilibrium point is linearly stable if following conditions are satisfied.

$$\begin{aligned} &2d + m - \beta_1 \overline{S}_1 > 0 \\ &2d + \gamma_1 + m - \beta_1 \overline{S}_1 - w > 0 \\ &2d - \gamma_1 - \mu + m > 0 \\ &2d - \beta_2 \overline{S}_2 - m > 0 \\ &2d + \gamma_2 - \beta_2 \overline{S}_2 - m - w > 0 \\ &2d - \gamma_2 - m > 0 \\ &\delta - w - \beta_1 \overline{S}_1 - \beta_2 \overline{S}_2 > 0 \end{aligned} \tag{13}$$

(ii) The endemic equilibrium point is linearly stable if following conditions are satisfied.

$$\begin{aligned} &2d + m + \beta_1 P^* - \beta_1 S_1^* > 0 \\ &2d + \gamma_1 + m - \beta_1 P^* - \beta_1 S_1^* - w > 0 \\ &2d - \gamma_1 - \mu + m > 0 \\ &2d - \beta_2 P^* - \beta_2 S_2^* - m > 0 \\ &2d + \gamma_2 - \beta_2 P^* - \beta_2 S_2^* - m - w > 0 \\ &2d - \gamma_2 - m > 0 \\ &\delta - w - \beta_1 S_1^* - \beta_2 S_2^* > 0 \end{aligned} \tag{14}$$

(iii) The disease free equilibrium point is non- linearly stable in the region given by

$$D = \{(S_1, I_1, R_1, S_2, I_2, R_2, P) : 0 < S_1 + I_1 + R_1 \le \alpha_1, 0 < S_2 + I_2 + R_2 \le \alpha_2 \ \& \ P > 0$$
$$\text{where, } \alpha_1 = \Lambda/(d+m) \ \& \ \alpha_2 = m\Lambda/d(d+m)\}$$

if following conditions are satisfied.

$$2d + m - \beta_1 \Lambda/(d+m) > 0$$

$$
\begin{aligned}
&2d+\gamma_1+m-\beta_1\Lambda/(d+m)-w>0\\
&2d-\gamma_1-\mu+m>0\\
&2d-\beta_2 m\Lambda/d(d+m)-m>0\\
&2d+\gamma_2-\beta_2 m\Lambda/d(d+m)-m-w>0\\
&2d-\gamma_2-m>0\\
&\delta-w-(\beta_2 m/d+\beta_1)\Lambda/(d+m)>0
\end{aligned}
\tag{15}
$$

(iv) The endemic equilibrium point is non-linearly stable in the region D if the following conditions are satisfied.

$$
\begin{aligned}
&2d+m+\beta_1 P^*-\beta_1\Lambda/(d+m)>0\\
&2d+\gamma_1+m-\beta_1 P^*-\beta_1\Lambda/(d+m)-w>0\\
&2d-\gamma_1-\mu+m>0\\
&2d+\beta_2 P^*-\beta_2 m\Lambda/d(d+m)-m>0\\
&2d+\gamma_2-\beta_2 P^*-\beta_2 m\Lambda/d(d+m)-m-w>0\\
&2d-\gamma_2-m>0\\
&\delta-w-(\beta_2 m/d+\beta_1)\Lambda/(d+m)>0
\end{aligned}
\tag{16}
$$

Now we analyse the main model given by (1) to (7)

2. Linear Stability Analysis:

2.1 Linear Stability Analysis of the Disease Free Equilibrium Point E_0

Consider the following transformation about the equilibrium E_0.

$$
\begin{aligned}
&S_1(t)=\overline{S}_1+n_1(t),\quad I_1(t)=n_2(t),\quad R_1(t)=\overline{R}_1+n_3(t),\\
&S_2(t)=\overline{S}_2+n_4(t),\quad I_2(t)=n_5(t),\quad R_2(t)=\overline{R}_2+n_6(t),\&\\
&P(t)=n_7(t)
\end{aligned}
$$

Using the above transformation in equations (1) to (7) we get

$$\frac{dn_1}{dt}=-(d+\mu)n_1(t)-mn_1(t-T)-\beta_1\{\overline{S}_1+n_1(t)\}n_7(t) \tag{17}$$

$$\frac{dn_2}{dt}=-(\gamma_1+d)n_2(t)-mn_2(t-T)+\beta_1\{\overline{S}_1+n_1(t)\}n_7(t) \tag{18}$$

$$\frac{dn_3}{dt}=\mu n_1(t)+\gamma_1 n_2(t)-dn_3(t)-mn_3(t-T) \tag{19}$$

$$\frac{dn_4}{dt}=mn_1(t-T)-dn_4(t)-\beta_2\{\overline{S}_2+n_4(t)\}n_7(t) \tag{20}$$

$$\frac{dn_5}{dt}=mn_2(t-T)-(\gamma_2+d)n_5(t)+\beta_2\{\overline{S}_2+n_4(t)\}n_7(t) \tag{21}$$

$$\frac{dn_6}{dt} = mn_3(t-T) + \gamma_2 n_5(t) - dn_6(t) \tag{22}$$

$$\frac{dn_7}{dt} = w\{n_2(t) + n_5(t)\} - \delta n_7(t) \tag{23}$$

Which after linearization becomes as:

$$\frac{dn_1}{dt} = -(d+\mu)n_1(t) - mn_1(t-T) - \beta_1 \overline{S}_1 n_7(t) \tag{24}$$

$$\frac{dn_2}{dt} = -(\gamma_1 + d)n_2(t) - mn_2(t-T) + \beta_1 \overline{S}_1 n_7(t) \tag{25}$$

$$\frac{dn_3}{dt} = \mu n_1(t) + \gamma_1 n_2(t) - dn_3(t) - mn_3(t-T) \tag{26}$$

$$\frac{dn_4}{dt} = mn_1(t-T) - dn_4(t) - \beta_2 \overline{S}_2 n_7(t) \tag{27}$$

$$\frac{dn_5}{dt} = mn_2(t-T) - (\gamma_2 + d)n_5(t) + \beta_2 \overline{S}_2 n_7(t) \tag{28}$$

$$\frac{dn_6}{dt} = mn_3(t-T) + \gamma_2 n_5(t) - dn_6(t) \tag{29}$$

$$\frac{dn_7}{dt} = w\{n_2(t) + n_5(t)\} - \delta n_7(t) \tag{30}$$

Now for the linear stability analysis of E_0 we proceed as follows:

Let, $$V_1 = \left[n_1(t) - \int_{t-T}^{t} mn_1(s)ds \right]^2 + (d + \mu + m + \beta_1 \overline{S}_1) \int_{t-T}^{t} \int_{s}^{t} mn_1^2(u)duds \tag{31}$$

Differentiating (31) with respect to t and using the inequality $(a^2 + b^2) \geq \pm 2ab$, we get

$$\frac{dV_1}{dt} \leq \{2(mT-1)(d+\mu+m) + (mT+1)\beta_1 \overline{S}_1\} n_1^2(t) + (mT+1)\beta_1 \overline{S}_1 n_7^2(t) \tag{32}$$

Let, $$V_2 = \left[n_2(t) - \int_{t-T}^{t} mn_2(s)ds \right]^2 + (d + \gamma_1 + m + \beta_1 \overline{S}_1) \int_{t-T}^{t} \int_{s}^{t} mn_2^2(u)duds \tag{33}$$

Differentiating (33) with respect to t and using the inequality $(a^2 + b^2) \geq \pm 2ab$, we get

$$\frac{dV_2}{dt} \leq \{2(mT-1)(d+\gamma_1+m) + (mT+1)\beta_1 \overline{S}_1\} n_2^2(t) + (mT+1)\beta_1 \overline{S}_1 n_7^2(t) \tag{34}$$

Let, $$V_3 = \left[n_3(t) - \int_{t-T}^{t} mn_3(s)ds \right]^2 + (d + \gamma_1 + m + \mu) \int_{t-T}^{t} \int_{s}^{t} mn_3^2(u)duds \tag{35}$$

Differentiating (35) with respect to t and using the inequality $(a^2 + b^2) \geq \pm 2ab$, we get

$$\frac{dV_3}{dt} \le \{2(mT-1)(d+m)+(mT+1)(\mu+\gamma_1)\}n_3^2(t)+(mT+1)\{\mu n_1^2(t)+\gamma_1 n_2^2(t)\} \tag{36}$$

Let, $V_4 = n_4^2(t) + \int_{t-T}^{t} m n_1^2(s)ds$ (37)

Differentiating (37) with respect to t and using the inequality $(a^2+b^2) \ge \pm 2ab$, we have

$$\frac{dV_4}{dt} \le mn_1^2(t)+(m-2d+\beta_2\overline{S}_2)n_4^2(t)+\beta_2\overline{S}_2 n_7^2(t) \tag{38}$$

Let, $V_5 = n_5^2(t) + \int_{t-T}^{t} m n_2^2(s)ds$ (39)

Differentiating (39) with respect to t and using the inequality $(a^2+b^2) \ge \pm 2ab$, we have

$$\frac{dV_5}{dt} \le mn_2^2(t)+(m-2\gamma_2-2d+\beta_2\overline{S}_2)n_5^2(t)+\beta_2\overline{S}_2 n_7^2(t) \tag{40}$$

Let, $V_6 = n_6^2(t) + \int_{t-T}^{t} m n_3^2(s)ds$ (41)

Differentiating (41) with respect to t and using the inequality $(a^2+b^2) \ge \pm 2ab$, we have

$$\frac{dV_6}{dt} \le mn_3^2(t)+(m+\gamma_2-2d)n_6^2(t)+\gamma_2 n_5^2(t) \tag{42}$$

Let, $V_7 = n_7^2(t)$ (43)

Differentiating (43) with respect to t and using the inequality $(a^2+b^2) \ge \pm 2ab$, we have

$$\frac{dV_7}{dt} \le w\{n_2^2(t)+n_5^2(t)\}+2(w-\delta)n_7^2(t) \tag{44}$$

Now, we define a Lyapunov functional

$$V = V_1+V_2+V_3+V_4+V_5+V_6+V_7 \tag{45}$$

Then, using the above results we get

$$\begin{aligned}\frac{dV}{dt} \le -[&\{2(1-mT)(d+\mu+m)-(1+mT)(\beta_1\overline{S}_1+\mu)-m\}n_1^2(t)\\ &+\{2(1-mT)(\gamma_1+d+m)-(1+mT)(\beta_1\overline{S}_1+\gamma_1)-m-w\}n_2^2(t)\\ &+\{2(1-mT)(d+m)-(1+mT)(\mu+\gamma_1)-m\}n_3^2(t)\\ &+\{2d-\beta_2\overline{S}_2-m\}n_4^2(t)+\{2d+\gamma_2-\beta_2\overline{S}_2-m-w\}n_5^2(t)\\ &+\{2d-\gamma_2-m\}n_6^2(t)+2\{(\delta-w)-(mT+1)\beta_1\overline{S}_1-\beta_2\overline{S}_2\}n_7^2(t)]\end{aligned} \tag{46}$$

Thus, the disease free equilibrium point E_0 is linearly stable if following conditions are satisfied:

$$
\begin{aligned}
&2(1-mT)(d+\mu+m)-(1+mT)(\beta_1\overline{S}_1+\mu)-m>0\\
&2(1-mT)(\gamma_1+d+m)-(1+mT)(\beta_1\overline{S}_1+\gamma_1)-m-w>0\\
&2(1-mT)(d+m)-(1+mT)(\mu+\gamma_1)-m>0\\
&2d-\beta_2\overline{S}_2-m>0\\
&2d+\gamma_2-\beta_2\overline{S}_2-m-w>0\\
&2d-\gamma_2-m>0\\
&\delta-w-(mT+1)\beta_1\overline{S}_1-\beta_2\overline{S}_2>0
\end{aligned}
\tag{47}
$$

4.2 Linear Stability Analysis of the Endemic Equilibrium Point E_1

Consider the following transformation about the equilibrium E_1

$$
\begin{aligned}
&S_1(t)=S_1^*+x_1(t),\quad I_1(t)=I_1^*+x_2(t),\quad R_1(t)=R_1^*+x_3(t)\\
&S_2(t)=S_2^*+x_4(t),\quad I_2(t)=I_2^*+x_5(t),\quad R_2(t)=R_2^*+x_6(t)\\
&P(t)=P^*+x_7(t)
\end{aligned}
$$

Using the above transformation in equations (1) to (7) we get

$$\frac{dx_1}{dt}=-(\beta_1P^*+d+\mu)x_1(t)-mx_1(t-T)-\beta_1\{S_1^*+x_1(t)\}x_7(t) \tag{48}$$

$$\frac{dx_2}{dt}=\beta_1P^*x_1(t)-(\gamma_1+d)x_2(t)-mx_2(t-T)+\beta_1\{S_1^*+x_1(t)\}x_7(t) \tag{49}$$

$$\frac{dx_3}{dt}=\mu x_1(t)+\gamma_1x_2(t)-dx_3(t)-mx_3(t-T) \tag{50}$$

$$\frac{dx_4}{dt}=mx_1(t-T)-(\beta_2P^*+d)x_4(t)-\beta_2\{S_2^*+x_4(t)\}x_7(t) \tag{51}$$

$$\frac{dx_5}{dt}=mx_2(t-T)+\beta_2P^*x_4(t)-(\gamma_2+d)x_5(t)+\beta_2\{S_2^*+x_4(t)\}x_7(t) \tag{52}$$

$$\frac{dx_6}{dt}=mx_3(t-T)+\gamma_2x_5(t)-dx_6(t) \tag{53}$$

$$\frac{dx_7}{dt}=w\{x_2(t)+x_5(t)\}-\delta x_7(t) \tag{54}$$

Which after linearization becomes as:

$$\frac{dx_1}{dt}=-(\beta_1P^*+d+\mu)x_1(t)-mx_1(t-T)-\beta_1S_1^*x_7(t) \tag{55}$$

$$\frac{dx_2}{dt}=\beta_1P^*x_1(t)-(\gamma_1+d)x_2(t)-mx_2(t-T)+\beta_1S_1^*x_7(t) \tag{57}$$

$$\frac{dx_3}{dt}=\mu x_1(t)+\gamma_1x_2(t)-dx_3(t)-mx_3(t-T) \tag{58}$$

$$\frac{dx_4}{dt} = mx_1(t-T) - (\beta_2 P^* + d)x_4(t) - \beta_2 S_2^* x_7(t) \tag{59}$$

$$\frac{dx_5}{dt} = mx_2(t-T) + \beta_2 P^* x_4(t) - (\gamma_2 + d)x_5(t) + \beta_2 S_2^* x_7(t) \tag{60}$$

$$\frac{dx_6}{dt} = mx_3(t-T) + \gamma_2 x_5(t) - dx_6(t) \tag{61}$$

$$\frac{dx_7}{dt} = w\{x_2(t) + x_5(t)\} - \delta x_7(t) \tag{62}$$

Now for the linear stability analysis of E_1 we proceed as follows:

Let,

$$U_1 = \left[x_1(t) - \int_{t-T}^{t} mx_1(s)ds \right]^2 + (d + \mu + m + \beta_1 P^* + \beta_1 S_1^*) \int_{t-T}^{t} \int_{s}^{t} mx_1^2(u)duds \tag{63}$$

$$U_2 = \left[x_2(t) - \int_{t-T}^{t} mx_2(s)ds \right]^2 + (d + \gamma_1 + m + \beta_1 P^* + \beta_1 S_1^*) \int_{t-T}^{t} \int_{s}^{t} mx_2^2(u)duds \tag{64}$$

$$U_3 = \left[x_3(t) - \int_{t-T}^{t} mx_3(s)ds \right]^2 + (d + \mu + m + \gamma_1) \int_{t-T}^{t} \int_{s}^{t} mx_3^2(u)duds \tag{65}$$

$$U_4 = x_4^2(t) + \int_{t-T}^{t} mx_1^2(s)ds \tag{66}$$

$$U_5 = x_5^2(t) + \int_{t-T}^{t} mx_2^2(s)ds \tag{67}$$

$$U_6 = x_6^2(t) + \int_{t-T}^{t} mx_3^2(s)ds \tag{68}$$

$$U_7 = x_7^2(t) \tag{69}$$

Differentiating (63) to (69) with respect to t and using the inequality $(a^2 + b^2) \geq \pm 2ab$, we get,

$$\frac{dU_1}{dt} \leq \{2(mT-1)(d + \mu + m + \beta_1 P^*) + (mT+1)\beta_1 S_1^*\} x_1^2(t) + \beta_1 S_1^* (mT+1) x_7^2(t) \tag{70}$$

$$\begin{aligned} \frac{dU_2}{dt} \leq & (mT+1)\beta_1 P^* x_1^2(t) + \{(2mT-1)(d + \gamma_1 + m) + (mT+1)(\beta_1 P^* + \beta_1 S_1^*)\} x_2^2(t) \\ & + \beta_1 S_1^* (mT+1) x_7^2(t) \end{aligned} \tag{71}$$

$$\frac{dU_3}{dt} \leq (mT+1)\mu x_1^2(t) + (mT+1)\gamma_1 x_2^2(t) + \{(mT+1)(\mu + \gamma_1) + 2(mT-1)(d+m)\} x_3^2(t) \tag{72}$$

$$\frac{dU_4}{dt} \le mx_1^2(t) + \{m - 2(\beta_2 P^* + d) + \beta_2 S_2^*\}x_4^2(t) + \beta_2 S_2^* x_7^2(t) \tag{73}$$

$$\frac{dU_5}{dt} \le mx_2^2(t) + \beta_2 P^* x_4^2(t) + \{m - 2\gamma_2 - 2d + \beta_2 P^* + \beta_2 S_2^*\}x_5^2(t) + \beta_2 S_2^* x_7^2(t) \tag{74}$$

$$\frac{dU_6}{dt} \le mx_3^2(t) + \gamma_2 x_5^2(t) + (m + \gamma_2 - 2d)x_6^2(t) \tag{75}$$

$$\frac{dU_7}{dt} \le w\{x_2^2(t) + x_5^2(t)\} + 2(w - \delta)x_7^2(t) \tag{76}$$

Now, we define a Lyapunov functional

$$U = U_1 + U_2 + U_3 + U_4 + U_5 + U_6 + U_7$$

Then, using the above results we get

$$\begin{aligned} \frac{dU}{dt} \le -[&\{2(1-mT)(d+m+\mu+\beta_1 P^*) - (1+mT)(\beta_1 S_1^* + \beta_1 P^* + \mu) - m\}x_1^2(t) \\ &+ \{2(1-mT)(\gamma_1 + d + m) - (1+mT)(\beta_1 P^* + \beta_1 S_1^* + \gamma_1) - m - w\}x_2^2(t) \\ &+ \{2(1-mT)(d+m) - (1+mT)(\mu + \gamma_1) - m\}x_3^2(t) \\ &+ \{2d + \beta_2 P^* - \beta_2 S_2^* - m\}x_4^2(t) + \{2d + \gamma_2 - \beta_2 P^* - \beta_2 S_2^* - m - w\}x_5^2(t) \\ &+ \{2d - \gamma_2 - m\}x_6^2(t) + 2\{\delta - w - (mT+1)\beta_1 S_1^* - \beta_2 S_2^*\}x_7^2(t)] \end{aligned} \tag{77}$$

Thus, the endemic equilibrium point E_1 is linearly stable if following conditions are satisfied:

$$\begin{aligned} &2(1-mT)(d+m+\mu+\beta_1 P^*) - (1+mT)(\beta_1 S_1^* + \beta_1 P^* + \mu) - m > 0 \\ &2(1-mT)(\gamma_1 + d + m) - (1+mT)(\beta_1 P^* + \beta_1 S_1^* + \gamma_1) - m - w > 0 \\ &2(1-mT)(d+m) - (1+mT)(\mu + \gamma_1) - m > 0 \\ &2d + \beta_2 P^* - \beta_2 S_2^* - m > 0 \\ &2d + \gamma_2 - \beta_2 P^* - \beta_2 S_2^* - m - w > 0 \\ &2d - \gamma_2 - m > 0 \\ &\delta - w - (mT+1)\beta_1 S_1^* - \beta_2 S_2^* > 0 \end{aligned} \tag{78}$$

3. Non-Linear Stability Analysis

3.1 Non-Linear Stability Analysis of the disease free equilibrium point E_0

First we construct a region D as follows:

$$D = \{(S_1, I_1, R_1, S_2, I_2, R_2, P) : 0 < S_1 + I_1 + R_1 \le \alpha_1, 0 < S_2 + I_2 + R_2 \le \alpha_2 \;\&\; P > 0 \quad \text{where, } \alpha_1 = \Lambda/(d+m) \;\&\; \alpha_2 = m\Lambda/d(d+m)\}$$

Let W_{11} is a positive definite function given by

$$W_{11} = \left[n_1(t) - \int_{t-T}^{t} mn_1(s)ds\right]^2 + \left[n_2(t) - \int_{t-T}^{t} mn_2(s)ds\right]^2 + \left[n_3(t) - \int_{t-T}^{t} mn_3(s)ds\right]^2 + n_4^2(t) + n_5^2(t) + n_6^2(t) + n_7^2(t) \tag{79}$$

Differentiating (79) with respect to t and using the inequality $(a^2 + b^2) \geq \pm 2ab$ in the region D we get

$$\frac{dW_{11}}{dt} \leq \{(mT-2)(d+\mu+m)+\beta_1\alpha_1+(mT+1)\}\mu n_1^2(t)$$
$$+\{(mT-2)(d+\gamma_1+m)+\beta_1\alpha_1+(mT+1)\gamma_1+w\}n_2^2(t)$$
$$\{(mT-2)(d+m)+\mu+\gamma_1\}n_3^2(t)+\{-2d+m+\beta_2\alpha_2\}n_4^2(t)$$
$$+\{-2d-\gamma_2+\beta_2\alpha_2+m\}n_5^2(t)+\{-2d+\gamma_2+m\}n_6^2(t)$$
$$+2\{(w-\delta)+(mT+1)\beta_1\alpha_1+\beta_2\alpha_2\}n_7^2(t)$$
$$+(d+\mu+m+\beta_1\alpha_1)\int_{t-T}^{t} mn_1^2(s)ds+(d+\gamma_1+m+\beta_1\alpha_1)\int_{t-T}^{t} mn_2^2(s)ds$$
$$+(d+\mu+\gamma_1+m)\int_{t-T}^{t} mn_3^2(s)ds+mn_1^2(t-T)+mn_2^2(t-T)+mn_3^2(t-T) \tag{80}$$

Let, $W_{12}=(d+\mu+m+\beta_1\alpha_1)\int_{t-T}^{t}\int_{s}^{t} mn_1^2(u)duds+(d+\gamma_1+m+\beta_1\alpha_1)\int_{t-T}^{t}\int_{s}^{t} mn_2^2(u)duds$

$$+(d+\mu+\gamma_1+m)\int_{t-T}^{t}\int_{s}^{t} mn_3^2(u)duds+\int_{t-T}^{t} mn_1^2(s)ds+\int_{t-T}^{t} mn_2^2(s)ds+\int_{t-T}^{t} mn_3^2(s)ds \tag{81}$$

Differentiating (81) with respect to t we have,

$$\frac{dW_{12}}{dt}=(d+\mu+m+\beta_1\alpha_1)mTn_1^2(t)-(d+\mu+m+\beta_1\alpha_1)\int_{t-T}^{t} mn_1^2(s)ds$$
$$+(d+\gamma_1+m+\beta_1\alpha_1)mTn_2^2(t)-(d+\gamma_1+m+\beta_1\alpha_1)\int_{t-T}^{t} mn_2^2(s)ds$$
$$+(d+\mu+m+\gamma_1)mTn_3^2(t)-(d+\mu+m+\gamma_1)\int_{t-T}^{t} mn_3^2(s)ds$$
$$+mn_1^2(t)-mn_1^2(t-T)+mn_2^2(t)-mn_2^2(t-T)+mn_3^2(t)-mn_3^2(t-T) \tag{82}$$

Now, we define a Lyapunov functional

$$W_1 = W_{11} + W_{12} \tag{83}$$

Then from (80) and (83) we get,

$$\frac{dW_1}{dt} \leq -[\{2(1-mT)(d+\mu+m)-(1+mT)(\beta_1\alpha_1+\mu)-m\}n_1^2(t)$$
$$+\{2(1-mT)(d+\gamma_1+m)-(1+mT)(\beta_1\alpha_1+\gamma_1)-m-w\}n_2^2(t)$$

$$+\{2(1-mT)(d+m)-(1+mT)(\gamma_1+\mu)-m\}n_3^2(t)$$
$$+\{2d-m-\beta_2\alpha_2\}n_4^2(t)+\{2d+\gamma_2-\beta_2\alpha_2-m-w\}n_5^2(t)$$
$$+\{2d-\gamma_2-m\}n_6^2(t)+2\{(\delta-w)-(mT+1)\beta_1\alpha_1-\beta_2\alpha_2\}n_7^2(t)] \quad (84)$$

Thus, disease free equilibrium point E_0 is non-linearly stable in the region D if following conditions are satisfied:

$$\begin{aligned}
&2(1-mT)(d+\mu+m)-(1+mT)\{\beta_1\Lambda/(d+m)+\mu\}-m>0\\
&2(1-mT)(d+\gamma_1+m)-(1+mT)\{\beta_1\Lambda/(d+m)+\gamma_1\}-m-w>0\\
&2(1-mT)(d+m)-(1+mT)(\gamma_1+\mu)-m>0\\
&2d-m-\beta_2 m\Lambda/d(d+m)>0\\
&2d+\gamma_2-\beta_2 m\Lambda/d(d+m)-m-w>0\\
&2d-\gamma_2-m>0\\
&\delta-w-\{(mT+1)\beta_1+\beta_2 m/d\}\Lambda/(d+m)>0
\end{aligned} \quad (85)$$

3.2 Non-Linear Stability Analysis of equilibrium point E_1

Let W_{21} is a positive definite function given by

$$W_{21}=\left[x_1(t)-\int_{t-T}^{t}mx_1(s)ds\right]^2+\left[x_2(t)-\int_{t-T}^{t}mx_2(s)ds\right]^2+\left[x_3(t)-\int_{t-T}^{t}mx_3(s)ds\right]^2$$
$$+x_4^2(t)+x_5^2(t)+x_6^2(t)+x_7^2(t) \quad (86)$$

Differentiating (86) with respect to t and using the inequality $(a^2+b^2)\geq\pm 2ab$ then in the region D we get,

$$\frac{dW_{21}}{dt}\leq\{(mT-2)(\beta_1P^*+d+\mu+m)+(mT+1)(\beta_1P^*+\mu)+\beta_1\alpha_1\}x_1^2(t)$$
$$+\{(mT-2)(d+\gamma_1+m)+\gamma_1(mT+1)+\beta_1P^*+\beta_1\alpha_1+w\}x_2^2(t)$$
$$+\{(mT-2)(d+m)+\gamma_1+\mu\}x_3^2(t)+\{-2d+m+\beta_2\alpha_2-\beta_2P^*\}x_4^2(t)$$
$$+\{-2d+m+\beta_2\alpha_2+\beta_2P^*-\gamma_2+w\}x_5^2(t)+\{-2d+m+\gamma_2\}x_6^2(t)$$
$$+2\{(w-\delta)+(mT+1)\beta_1\alpha_1+\beta_2\alpha_2\}x_7^2(t)$$
$$+(\beta_1P^*+d+\mu+m+\beta_1\alpha_1)\int_{t-T}^{t}mx_1^2(s)ds+(\beta_1P^*+d+\gamma_1+m+\beta_1\alpha_1)\int_{t-T}^{t}mx_2^2(s)ds$$
$$+(\mu+d+\gamma_1+m)\int_{t-T}^{t}mx_3^2(s)ds+mx_1^2(t-T)+mx_2^2(t-T)+mx_3^2(t-T) \quad (87)$$

Let, $W_{22} = (\beta_1 P^* + d + \mu + m + \beta_1\alpha_1)\int_{t-T}^{t}\int_{s}^{t} m x_1^2(u)duds + \int_{t-T}^{T} m x_1^2(s)ds$

$$+(\beta_1 P^* + d + \gamma_1 + m + \beta_1\alpha_1)\int_{t-T}^{t}\int_{s}^{t} m x_2^2(u)duds + \int_{t-T}^{T} m x_2^2(s)ds$$

$$+(\mu + d + \gamma_1 + m)\int_{t-T}^{t}\int_{s}^{t} m x_3^2(u)duds + \int_{t-T}^{T} m x_3^2(s)ds \tag{88}$$

Differentiating (88) with respect to t ,we have

$$\frac{dW_{22}}{dt} = (\beta_1 P^* + d + \mu + m + \beta_1\alpha_1)\left[mTx_1^2(t) - \int_{t-T}^{T} m x_1^2(s)ds\right]$$
$$+(\beta_1 P^* + d + \gamma_1 + m + \beta_1\alpha_1)\left[mTx_2^2(t) - \int_{t-T}^{T} m x_2^2(s)ds\right]$$
$$+(\gamma_1 + d + \mu + m)\left[mTx_3^2(t) - \int_{t-T}^{T} m x_3^2(s)ds\right]$$
$$+x_1^2(t) + x_2^2(t) + x_3^2(t) - x_1^2(t-T) - x_2^2(t-T) - x_3^2(t-T) \tag{89}$$

Now, we define a Lyapunov functional

$$W_2 = W_{21} + W_{22} \tag{90}$$

Then from (88) and (90) we get,

$$\begin{aligned}\frac{dW_2}{dt} \le -[&\{2(1-mT)(\beta_1 P^* + d + \mu + m) - (1+mT)(\beta_1 P^* + \beta_1\alpha_1 + \mu) - m\}x_1^2(t)\\ &+\{2(1-mT)(\gamma_1 + d + m) - (1+mT)(\beta_1 P^* + \beta_1\alpha_1 + \gamma_1) - m - w\}x_2^2(t)\\ &+\{2(1-mT)(d+m) - (1+mT)(\gamma_1 + \mu) - m\}x_3^2(t)\\ &+\{2d + \beta_2 P^* - \beta_2\alpha_2 - m\}x_4^2(t) + \{2d + \gamma_2 - \beta_2 P^* - \beta_2\alpha_2 - m - w\}x_5^2(t)\\ &+\{2d - m - \gamma_2\}x_6^2(t) + 2\{\delta - w - \beta_2\alpha_2 - (1+mT)\beta_1\alpha_1\}x_7^2(t)\end{aligned} \tag{91}$$

Thus endemic equilibrium point E_1 is non-linearly stable in the region D if following conditions are satisfied.

$$2(1-mT)(\beta_1 P^* + d + \mu + m) - (1+mT)(\beta_1 P^* + \beta_1\Lambda/(d+m) + \mu) - m > 0$$
$$2(1-mT)(\gamma_1 + d + m) - (1+mT)(\beta_1 P^* + \beta_1\Lambda/(d+m) + \gamma_1) - m - w > 0$$
$$2(1-mT)(d+m) - (1+mT)(\gamma_1 + \mu) - m > 0$$
$$2d + \beta_2 P^* - \beta_2 m\Lambda/d(d+m) - m > 0 \tag{92}$$

$$2d + \gamma_2 - \beta_2 P^* - \beta_2 m\Lambda / d(d+m) - m - w > 0$$

$$2d - m - \gamma_2 > 0$$

$$\delta - w - \{\beta_2 m / d + (1+mT)\beta_1\}\Lambda /(d+m) > 0$$

Discussion

Here we have analyzed an ordinary differential equation model with time delay for transmission of infectious diseases by droplet infection. By conducting the linear and non-linear stability analysis of the disease free and endemic equilibrium points, it has been shown that disease free and endemic equilibrium points are stable under the conditions involving disease related parameters and time delay. It can be observed from the conditions of stability that if T becomes zero then the stability conditions given for model-1 coincide with the conditions given for model 1(b) which is a special case of model-1. From the values of equilibrium points it may be observed that if the rate of vaccination increases then equilibrium level of disease will decrease. From the stability conditions of model-1 it may be observed that the delay in maturation may lead to the instability of the otherwise stable equilibrium points for large time-delay.

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