ASCENT AND DESCENT OF PRODUCT AND SUM OF TWO COMPOSITION OPERATORS ON ℓ^p SPACES

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Abstract

Let l^p $(1 \leq p \leq \infty)$ be the Banach space of all p-summable sequences (bounded sequences for $p = \infty$) of complex numbers under the standard p-norm on it and $C\phi$ be a composition operator on l^p induced by a function ϕ on N into itself. In this paper we discuss the ascent and descent of Product and Sum of two composition operators on l^p spaces.

Introduction

Let X denote an arbitrary vector space and T be a linear operator on X. Let D(T), N(T) and R(T) denote domain, kernel and range of T respectively. Let N denote the set of natural numbers. The following statements and definitions are relevant and instructive in our context.

Theorem 1.1. $N(T^n) \subseteq N(T^{n+1})$; $n = 0, 1, 2, \ldots$ If $N(T^k) = N(T^{k+1})$ for some k, then $N(T^k) = N(T^n)$ when $n \geq k$.

Definition 1.1. If there is some integer $n \geq 0$ such that $N(T^n) = N(T^{n+1})$, the smallest such integer is called the ascent of T and is denoted by a(T). If no such integer exists we say that $a(T) = \infty$.

Theorem 1.2. $R(T^{n+1}) \subseteq R(T^n)$; $n = 0, 1, 2, \ldots$ If $R(T^{k+1}) = R(T^k)$ for some k, then $R(T^n) = R(T^k)$ when $n \geq k$.

Definition 1.2. If there is some integer $n \geq 0$ such that $R(T^{n+1}) = R(T^n)$, the smallest such integer is called the descent of T and is denoted by d(T). If no such integer exists we say that $d(T) = \infty$.

Theorem 1.3. If a(T) is finite and d(T) = 0, then a(T) = 0.

Theorem 1.4. If a(T) and d(T) are both finite, then necessarily $a(T) \leq d(T)$.

Theorem 1.5. If D(T) = X and the ascent and descent of T are both finite, they are equal.

Composition operators on l^p spaces

The composition operator $C\phi$ on l^p induced by a function ϕ on N into itself is defined by $C_\phi(f) = f.\phi$ for all $f \in l^p$. It is well known that a necessary and sufficient condition for a function ϕ on N into itself to induce a composition operator on l^p is that the set $\{ |\phi^{-1}(n)| : n \}$ is bounded. Here $|\phi^{-1}(n)|$ denotes the number of elements in $\phi^{-1}(n)$; (see [20] and [24]).

The study of ascent and descent of an operator has been done as a part of spectral properties of an operator (see [1], [2], [5] and [6]). Since composition operators provide diverse and illuminating examples of operators which leads to useful insight into structure theory of operators, it is desirable to study ascent and descent of these operators and their sum and product.

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Examples 2

We now give different examples of product of two composition operators which is given below:

Example 2.1. Let ϕ be a self-map on defined as follows.

$$\phi(n) = \begin{cases} 1, & if \ n = 1.2 \\ n-1, & if \ n > 2 \end{cases}$$

Here a(Cϕ) = 0 and d(Cϕ) = ∞. Clearly CϕCφ ≠ CφCϕ
Let φ be a self-map on N defined as follows.

$$\varphi(n) = \begin{cases} n, & if \ n \ is \ odd \\ n-1, & if \ n \ is \ even. \end{cases}$$

Here a(Cφ) = 1 = d(Cφ). Clearly Cϕ Cφ ≠ CφCϕ.
But a(Cϕ Cφ) = 1 = d(Cϕ Cφ) and a(CφCϕ) = 1 = d(CφCϕ).
Ascent and Descent of Sum and Product of two Composition Operators

Example 2.2. Let ϕ be a self-map on **N** defined as follows.

$$\phi\ (n) = \begin{cases} n, & if \ n \ is \ odd \\ n-1, & if \ n \ is \ even. \end{cases}$$

Here a(Cϕ) = 1= d(Cϕ).
Let φbe a self-map on N defined as follows.

$$\varphi\ (n) = \begin{cases} 1, & if \ n = 1.2 \\ 2, & if \ n = 3 \\ n+1 & if \ n > 3. \end{cases}$$

Here a(Cφ) = ∞ = d(Cφ). Clearly Cϕφ ≠ CφCϕ
But a(CϕCφ) = 2 = d(CϕCφ) and a(CφCϕ) =1= d(CφCϕ).

Example 2.3. Let ϕ be a self-map on N defined as follows.

$$\phi\ (n) = \begin{cases} 1, & if \ n = 1.2 \\ 2, & if \ n = 3 \\ n+1, & if \ n \in \{5.7.9......\} \\ n-1, & if \ n \in \{4.6.8......). \end{cases}$$

Here a(Cϕ) = 3 = d(Cϕ).
Let φbe a self-map on N defined as follows.

$$\varphi(n) = n + 2, \text{ if n is odd}$$

and

$$\varphi(2n\text{-}2) = \varphi(n) = n, \text{ if } n \in \{2,4,6,8,10........\}.$$

Here a(Cφ) = ∞ = d(Cφ). Clearly CϕCφ ≠ CφCϕ

But a(CϕCφ) = 1 = d(CϕCφ) and a(CφCϕ) =1= d(CφCϕ).

Example 2.4. Let ϕ be a self-map on N defined as follows.

$$\phi(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

Here a(Cϕ) = 1 = d(Cϕ).

Let φ be a self-map on N defined as follows.

$$\phi(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

Here a(Cφ) = 0 = d(Cφ). Clearly CϕCφ≠ CφCϕ

But a(CϕCφ) = 1 = d(CϕCφ) and a(CφCϕ) =1= d(CφCϕ).

Example 2.5. Let ϕ be a self-map on N defined as follows.

$$\phi(s) = s - 1, \text{if } s \in \{2,3,\ldots\ldots\ldots,n\}$$

and

$$\phi(t) = t, \text{ if } t \in \mathbf{N} - \{2,3,\ldots\ldots\ldots,n\}.$$

Here a(Cϕ) = 0 = d(Cϕ).

Let φbe a self-map on N defined as follows.

$$\varphi(t) = t, \ \forall t \in N$$

Here a(Cφ) = 0 = d(Cφ). Clearly CϕCφ = CφCϕ

But a(CϕCφ) = 0 = d(CϕCφ) and a(CφCϕ) = 0 = d(CφCϕ).

3. RESULTS

In this section we give a characterization of product of two composition operators on l^p spaces.

Theorem 3.1. Let Cϕ and Cφ be two composition operators on l^p spaces and CϕCφ= CφCϕ. Then the following results hold.

(i) a(CϕCφ) ≤ Max {a(Cϕ), a(Cφ)} (ii) d(CϕCφ) ≤ Max {d(Cϕ), d(Cφ)}.

Proof. (i) Case-I: If a(Cϕ) = ∞ or a(Cφ) = ∞. Then result (i) is obviously true.

Case-II: If a(Cϕ) = m < ∞ and a(Cφ) = n < ∞. This implies that $R(\phi^m) = R(\phi^{m+1})$ and $R(\varphi^n) = R(\varphi^{n+1})$. Now suppose n > m, Max {m, n} = n. We claim that $N((C\phi C\varphi)^{n+1}) = N((C\phi C\varphi)^n)$. Let f $\in N((C\phi C\varphi)^{n+1})$. This implies that $(C\phi C\varphi)^{n+1}(f) = 0$. Since CϕCφ = CφCϕ.

then $(C\phi^{n+1} C\varphi^{n+1})(f) = 0$.This implies that $f(\varphi^{n+1}\phi^{n+1}) = 0$. Since $R(\phi^{m}) = R(\phi^{m+1})$ and $R(\varphi^{n}) = R(\varphi^{n+1})$. Hence $f(\varphi^{n}\phi^{n}) = 0$. Therefore $(C\phi C\varphi)^{n}(f) = 0$. Thus $f\in N((C\phi C\varphi)^{n})$.Thus

$$N((C\phi C\varphi)^{n+1}) \subseteq N((C\phi C\varphi)^{n}). \quad (1)$$

Since it is obvious that

$$N((C\phi C\varphi)^{n}) \subseteq N((C\phi C\varphi)^{n+1}). \quad (2)$$

Combining equations (1) and (2), we get

$$N((C\phi C\varphi)^{n+1}) = N((C\phi C\varphi)^{n}).$$

Hence $a(C\phi C\varphi) \leq n = \text{Max}\{a(C\phi), a(C\varphi)\}$.

(ii) Case-I: If $d(C\phi) = \infty$ or $d(C\varphi) = \infty$. Then result (ii) is obviously true.
Case-II: If $d(C\phi) = m < \infty$ and $d(C\varphi) = n < \infty$. This implies that $\phi : R(\phi^{m}) \to R(\phi^{m})$ is injective and $\varphi : R(\varphi^{m}) \to R(\varphi^{m})$ is injective. Since $C\phi C\varphi = C\varphi C\phi$ then $\phi : R(\phi^{m+i}) \to R(\phi^{m+i})$ is injective and $\varphi : R(\varphi^{m+i}) \to R(\varphi^{m+i})$ is injective for each $i \geq 1$. Now suppose $n > m$, $\text{Max}\{m, n\} = n$. We claim that $(\phi\varphi) : R((\phi\varphi)^{n}) \to R((\phi\varphi)^{n})$ is injective. Let $k_1 \in R(\phi^{n})$ and $k_2 \in R(\varphi^{n})$ such that $k_1 \neq k_2$. It is given that $\phi : R(\phi^{n}) \to R(\phi^{n})$ is injective and $\varphi : R(\varphi^{n}) \to R(\varphi^{n})$ is injective. This implies that $\phi(k_1) \neq \phi(k_2)$ and $\varphi(k_1) \neq \varphi(k_2)$. Thus $(\phi\varphi)(k_1) \neq (\phi\varphi)(k_2)$. Therefore $(\phi\varphi) : R((\phi\varphi)^{n}) \to R((\phi\varphi)^{n})$ is injective.

Hence $d(C\phi C\varphi) \leq n = \text{Max}\{a(C\phi), a(C\varphi)\}$. □

Theorem 3.2. $a(C\phi C\varphi) = \infty$ if and only if there exists a sequence of distinct integers $\{n_k\}$ such that $n_k \notin R((\varphi\phi)^{k})$ but $n_k \in R((\varphi\phi)^{k-1})$ for each $k \geq 1$.

Proof. $C\phi C\varphi = C\phi.\varphi$ is a composition operator induced by $\varphi.\phi$ Now by theorem 3.1in [5] follows that $a(C\phi C\varphi) = \infty$ if and only if there exists a sequence of distinct integers $\{nk\}$ such that $n_k \notin R((\varphi.\phi)^{k})$ but $n_k \in R((\varphi.\phi)^{k-1})$ for each $k \geq 1$. □

Theorem 3.3. $d(C\phi C\varphi) = \infty$ if and only if $(\varphi.\phi) : R((\varphi.\phi)^{k}) \to R((\varphi.\phi)^{k})$ is not one-to-one for all $k \geq 0$.

Proof. $C\phi C\varphi = C\varphi.\phi$ is a composition operator induced by $\varphi.\phi$. Now by theorem 3.2 in [5] follows that $d(C\phi C\varphi) = \infty$ if and only if $(\varphi.\phi) : R((\varphi.\phi)^{k}) \to R((\varphi.\phi)^{k})$ is not one-to-one for all $k \geq 0$. □

Theorem 3.4. $a_e(C\phi C\varphi) = \infty$ if and only if there exist a sequence $\{E_k\}^{\infty}k=1$ of subsets of **N** such that each E_k is infinite, $E_k \subseteq R((\varphi.\phi)^{k-1})$ and $R((\varphi.\phi)^k) \cap E_k = \phi$ for each $k \in$ **N**. □

Proof. $C\phi C\varphi = C\varphi.\phi$ is a composition operator induced by$\varphi.\phi$. Now by theorem 3.1in [6] follows that $a(C\phi C\varphi) = \infty$ if and only if there exist a sequence $\{E_k\}^{\infty}_{k=1}$ of subsets of **N** such that each E_k is infinite, $E_k \subseteq R((\varphi.\phi)^{k-1})$ and $R((\varphi.\phi)^k) \cap E_k = \phi$ for each $k\in$ **N**. □

Theorem 3.5. $d_e(C\phi C\varphi) = \infty$ if and only if for each $k \geq 0$; $|\phi^{-1}(n)| > 1$ for infinitely many $n\in R((\varphi.\phi)^k)$.

Proof. $C\phi C\varphi = C\varphi.\phi$ is a composition operator induced by $\varphi.\phi$. Now by theorem 3.2 in [6] follows that $de(C\phi C\varphi) = \infty$ if and only if for each $k \geq 0$; $|\phi^{-1}(n)| > 1$ for infinitely many $n\in R((\varphi.\phi)^k)$. □

4. Example

We now give different examples of sum of two composition operators which is given below :

Example 4.1. Let ϕ be a self-map on **N** defined as follows.

$$\phi(1) = 1 = \phi(2)$$

and

$$\phi(n) = n, \forall n \geq 3.$$

Here $a(C\phi) = 1 = d(C\phi)$.

Let φ be a self-map on **N** defined as follows.

$$\varphi(3) = 3 = \varphi(4)$$

and

$$\varphi(n) = n, \text{ if } n\in \mathbf{N}-\{3, 4\}.$$

Here $a(C\varphi) = 1 = d(C\varphi)$.

But $a(C\phi + C\varphi) = 0 = d(C\phi + C\varphi)$.

Example 4.2. Let ϕ be a self-map on **N** defined as follows.

$$\phi(1) = 1 = \phi(2)$$

and

$$\phi(n) = n\text{-}1, \forall n \geq 3.$$

Here a(Cϕ) = 0 and d(Cϕ) = ∞.

Let φ be a self-map on **N** defined as follows.

$$\varphi(n) \quad \begin{matrix} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even}. \end{matrix}$$

Here a(Cφ) = 1 = d(Cφ).

But a(Cϕ + Cφ) = 0 and d(Cϕ + Cφ) = ∞.

Example 4.3. Let ϕ be a self-map on **N** defined as follows.

$$\phi(n) \quad \begin{matrix} n \quad 1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even}. \end{matrix}$$

Here a(Cϕ) = 0 = d(Cϕ).

Let φ be a self-map on **N** defined as follows.

$$\varphi(n) \quad \begin{matrix} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even}. \end{matrix}$$

Here a(Cφ) = 1 = d(Cφ).

But a(Cϕ + Cφ) = 0 = d(Cϕ + Cφ).

Example 4.4. Let ϕ be a self-map on **N** defined as follows.

ϕ(n) = 2n-1, for each natural number.

Here a(Cϕ) = ∞ and d(Cϕ) = 0.

Let φ be a self-map on defined as follows.

$$\varphi(n) = 2n, \text{ for each natural number.}$$

Here a(Cφ) = ∞ and d(Cφ) = 0.

But a(Cϕ+ Cφ) = ∞ and d(Cϕ + Cφ) = 0.

Example 4.5 Let ϕ be a self-map on **N** defined as follows.

$$\phi(1) = 1 = \phi(2)$$

and

$$\phi(n) = n, n \ \forall \geq 3.$$

Here a(Cϕ) = 1 = d(Cϕ).

Let φ be a self-map on **N** defined as follows.

$$\varphi(n) \quad \begin{matrix} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even}. \end{matrix}$$

Here a(Cφ) = 1 = d(Cφ).
But a(Cϕ + Cφ) = 1 = d(Cϕ + Cφ).

Example 4.6 Let ϕ be a self-map on **N** defined as follows.

$$\phi(n) \quad \begin{matrix} n \quad 1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even}. \end{matrix}$$

Here a(Cφ) = 0 = d(Cφ).
Let φ be a self-map on **N** defined as follows.

φ(n) = n+1, for each natural number.

Here a(Cφ) = ∞ and d(Cφ) = 0.
But a(Cϕ + Cφ) = 0 = d(Cϕ + Cφ).

5. RESULTS

In this section we study ascent and descent of sum of two composition operators on l^p spaces.

Theorem 5.1. If ϕ or φ is an injective self-map on **N** then d(Cϕ+Cφ) = 0.
Converse is not true.

Proof. If ϕ is injective then R(Cϕ) = l^p. Now R(Cϕ + Cφ) = R(Cϕ) + R(Cφ) = l^p + R(Cφ) = l^p. Thus d(Cϕ+Cφ) = 0. The following example shows that the converse is not true. □

Example 5.1. Let ϕ be a self-map on **N** defined as follows.

ϕ(1) = 1 = ϕ(2)

and

ϕ(n) = n, for each n ≥ 3.

Let φ be a self-map on **N** defined as follows.

φ(3) = 3 = φ(4)

and

φ(n) = n, if n ∈ **N** {3, 4}.

Then d(Cϕ + Cφ)= 0.

Theorem 5.2. If ϕ and φ be any two self-maps on **N** into itself and let Cϕ+φ denote the composition operator induced by ϕ + φ. Then a(Cϕ+φ) = ∞.

Proof. Let n_1 = 1. For k ≥ 2, let n_k denote the smallest element of R($(\phi+\varphi)^{k-1}$).
Then clearly we get a sequence $\{n_k\}^{\infty}_{k=1}$ such that n_k ∈ R($(\phi + \varphi)^{k-1}$)) but n_k ∉ R($(\phi+\varphi)^k$). Hence by theorem 3.2 of [5] it follows that a(Cϕ+φ) = ∞.

Theorem 5.3. $d(C\phi + \varphi) = \infty$ if and only if $(\phi + \varphi) : R((\phi + \varphi)^k) \to R((\phi + \varphi)^k)$ is not one-to-one for all $k \geq 0$.

Proof. Follows from theorem 3.2 of [5]. □

Remark 5.1. Let ϕ and φ are both injective neither of them is surjective. If $\phi \neq \varphi$ then clearly $a(C\varphi + C\varphi) = \infty$.

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