SOME METHODS OF CONSTRUCTIONS OF REGULAR BLOCK DESIGNS

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Abstract

Some methods of construction of regular group divisible designs are described which lead to some new series of designs. In particular, some new non-isomorphic solutions for certain Group divisible (GD) designs are presented.

AMS Subject classifications

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Keywords and Phrases

Regular GD design; BIB design; non-isomorphic.

Introductions

Group divisible (GD) designs constitute the largest, simplest and perhaps the most important type of two-associate partially balanced incomplete block designs. A GD design is an arrangement of $v = mn$ treatments in b blocks each of size $k<v$ distinct treatments; each treatment is replicated r times and the set of treatments can be partitioned in $m \geq 2$ groups of $n \geq 2$ treatments each, any two distinct treatments (θ, φ) occurring together in λ_1 blocks if they belong to the same group and in λ_2 blocks if they belong to different groups. Furthermore, if $r-\lambda_1=0$, the GD design is said to be singular, if $r-\lambda_1>0$ and $rk-v\lambda_2=0$ it is called semi-regular(SR); and if $r-\lambda_1>0$ and $rk-v\lambda_2>0$, it is called regular(R).

For the GD designs, it holds that

$$(rk-v\lambda_2)-(\tau-\lambda_1)=n(\lambda_1-\lambda_2) \qquad \text{............................ (1.1)}$$

Where $rk-v\lambda_2(=\theta_1, say)$ and $r-\lambda_1(=\theta_2, say)$ are eigen-values of NN' other than rk, with the respective multiplicities (m-1) and m (n-1), and then $\theta_1=\geq 0$ and $\theta_2=\geq 0$. Note that N is the incidence matrix of GD designs.

The regular type means that all the eigen values of NN' are positive. For a regular block design, Fisher's inequality, i.e. the number of blocks being bounded below by the number of treatments always holds.

It follows from (1.1) that if $|\theta_1-\theta_2|=1$ then the GD design does not exist. Hence $|\theta_1-\theta_2|=\geq 2$ Furthermore, if $|\theta_1-\theta_2|$ is a prime, then $\lambda_2=\lambda_1\pm 1$. Note that in an SGD design $\lambda_2<\lambda_1$ and in an SRGD design $\lambda_2>\lambda_1$. It seems that smaller values of eigen values θ_1 and θ_2 yield more efficient GD designs in the sense of providing good estimates for certain functions of treatment effects depending on association in treatment structure.

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Clalworthy (1973) gave a table of 209 regular GD designs. Since, then Freeman (1976), Kageyama and Tanaka (1981), Dey(1977), Bhagwandas and Parihar (1980,1982), Dey and Nigam (1985), Bhagwandas et al. (1895), Sinha and Kageyama (1986), and Sinha (1987) have given several methods of constructing GD designs.

In this paper, we describe some new methods of construction of GD design, thereafter some new series of GD designs are obtained. In particular, non-isomorphic solutions of certain GD designs with $r,k \leq 10$ are reported.

THEORAM 2.1. If N denotes the incidence matrix of a BIB design *(v = 2k, b, r, k, λ)*, then the incidence structure

$$S = \begin{bmatrix} 0 & 0 & N & N & N & J\text{-}N & N & N \\ N & N & 0 & 0 & N & N & N & J\text{-}N \\ N & J\text{-}N & N & N & 0 & 0 & J\text{-}N & J\text{-}N \\ J\text{-}N & J\text{-}N & J\text{-}N & N & N & N & 0 & 0 \end{bmatrix} \qquad \text{.............. (2.1)}$$

is the incidence matrix of a regular GD design with parameters

$$v^* = 4v,\ b^* = 8b,\ r^* = 6r,\ k^* = 3k,\ \lambda_1^* = 6\lambda,\ \lambda_2^* = 2r,$$

Proof. Clearly S is of order $v^* \times b^*$ and has row (column) sum $r^*(k^*)$. Further, let $P = SS'$ (S' begin the transpose of S). Then we have,

$$P = I_4 \otimes (A - B) + J_4 \otimes B, \qquad \text{............................ (2.2)}$$

$$\text{Where } A = 5NN' + (J - N)(J - N)',$$

$$= 6(R - \lambda)\, I_v + 6\lambda J_v \qquad \text{............................ (2.3)}$$

$$B = 2NN' + (J - N)N' + N(J - N)',$$

$$= 2rJ_v \qquad \text{............................ (2.4)}$$

From (2.3) and (2.4), it follows that $\lambda^*_1 = 6\lambda$ and $\lambda^*_2 = 6r$. The mathematical form of P shows that S is the incidence matrix of a GD design. Since for resulting GD design it holds that $r^* - \lambda^*_1 > 0$ and $r^* k^* - v^* \lambda^*_2 > 0$; hence this is a regular GD design.

The applications of Theorem 2.1, Consider N, the incidence matrix of BIBD (2,2,1,1,0), we obtain regular GD design with parameters $v^* = 8, v^* = 16, r^* = 6, k^* = 3, \lambda^*_1 = 0, \lambda^*_2 = 2, m = 4, n = 2$. The same design is listed as R55 in Clatworthy (1973). The solution obtain by us is new non-isomorphic to that given in Clatworthy (1973), in both the solutions the block intersections numbers are different. In our solution the blocks are as follows:

(3,5,8), (4,6,7), (3,6,8), (4,5,7), (1,5,8), (2,6,7),

(1,5,7), (2,6,8), (1,3,7), (2,4,8), (2,3,7), (1,4.8),

(1,3,6), (2,4,5), (1,4,6),(2,3,5).

Remark 2.1.

The complementary design of RGD design $v^* = 8, v^* = 16, r^* = 6, k^* = 3, \lambda^*_1 = 0, \lambda^*_1 = 2, m = 4, n = 2$. has the parameters $v^* = 8, v^* = 16, r^* = 10, k^* = 5, \lambda_1 = 4, \lambda_2 = 6, m = 4, n = 2$. This is R136 in in Clatworthy(1973), but the present solution is non-isomorphic to the reported solution in Clatworthy(1973) in terms of block structure. The blocks in the present solution are as follows:

(1,2,4,6,7),	(1,2,3,5,8),	(1,2,4,5,7),	(1,2,3,6,8),
(2,3,4,6,7),	(1,3,4,5,8),	(2,3,4,6,8),	(1,3,4,5,7),
(2,4,5,6,8),	(1,3,5,6,7),	(1,4,5,6,8),	(2,3,5,6,7),
(2,4,5,7,8),	(1,3,6,7,8),	(2,3,5,7,8),	(1,4,6,7,8),

THEOREM 2.2. If N be the incidence matrix of a BIB design with parameters (v, b, r, k, λ) and $v = 2k$, then

	N	N	N	N	N	N	N	N	N	N	0	0	0	0	0
	N	N	N	J-N	J-N	J-N	0	0	0	0	N	N	N	N	0
S=	N	N	0	N	0	0	J-N	J-N	J-N	0	N	J-N	J-N	0	N
	N	0	J-N	0	J-N	0	N	N	0	J-N	J-N	J-N	0	N	N
	0	0	N	J-N	0	J-N	N	O	J-N	N	J-N	O	J-N	J-N	N
	0	N	0	0	N	J-N	0	N	J-N	J-N	0	J-N	N	J-N	J-N

Yield a regular GD design with parameters

$$v^* = 6v,\ b^* = 15b,\ r^* = 10r,\ k^* = 4k,\ \lambda^*_1 = 10\lambda,\ \lambda^*_2 = 3r,\ m = 6,\ n = v$$

Proof : Obvious.

As an illustration, Consider N, the incidence matrix of BIBD (2, 2, 1, 1, 0) in Theorem 2.2, we obtain regular GD design with parameters $v^* = 12, b^* = 30, r^* = 10, k^* = 4, \lambda^*_1 = 10\lambda, \lambda^*_2 = 3, m = 6, n = v$ The same RGD is reported by Freelman (1976) as R110b. Solutions obtained by us is non-isomorphic to that reported in Freeman (1976). The blocks in our solutions are as follows:

(1,3,5,7),	(2,4,6,8),	(1,3,5,11),	(2,4,6,12),	(1,3,8,9),
(2,4,7,10),	(1,4,5,10),	(2,3,6,9),	(1,4,8,11),	(2,3,7,11),
(1,4,10,12),	(2,3,9,11),	(1,6,7,9),	(2,5,8,10),	(1,6,10,12),

(2,5,9,11),	(1,8,9,12),	(2,7,10,11),	(3,5,8,10),	(4,6,7,9),
(3,6,8,12),	(4,5,7,11),	(3,6,10,11),	(4,5,9,12),	(3,7,10,12),
(4,8,9,11),	(5,7,9,12),	(6,8,10,11),	(1,6,7,11),	(2,5,8,12),

THEOREM 2.3.

The existence of a BIB design with $v = 2k$, implies the existence of a regular GD design with parameters

$$v^* = 7v, b^* = 14b,\ r^* = 8r, k^* = 4k, \lambda_1^* = 8\lambda,\ \lambda_2^* = 2r,\ m = 7,\ n = v$$

Proof : We start with the BIB Design $v = 2k, b = 2r, r, k, \lambda$. Now we form the incidence structure S by arranging N the incidence matrix of BIB Design, J-N, the complementary structure of N, and o the null matrix as

S =

N	N	N	N	N	N	N	N	0	0	0	0	0	0
N	0	0	J-N	N	J-N	0	0	0	J-N	N	0	N	N
N	0	N	0	J-N	0	J-N	0	N	0	J-N	N	0	N
0	0	J-N	N	0	0	0	J-N	J-N	N	0	N	N	0
0	N	J-N	0	0	J-N	N	0	N	N	O	0	J-N	N
0	N	0	J-N	0	0	J-N	N	0	N	N	N	0	J-N
0	N	0	0	J-N	N	0	J-N	N	0	N	J-N	N	0

is the incidence matrix of the required regular GD design.

As an illustration, Consider N as the incidence matrix of BIBD (2, 2, 1, 1, 0) in Theorem 2.3, the resulting regular GD design has the parameters

$v^* = 14,\ b^* = 28,\ r^* = 8,\ k^* = 4,\ \lambda_1^* = 0,\ ^*\lambda_2^* = 2;\ m = 7,\ n = 2,$ ……………(2.5)

This is deign R_{113} in Clatworthy (1973). Solution obtained by us is new non-

isomorphic to the solution reported in Clatworthy (1973). In our solutions the blocks are as follows:

(1,3,5,7),	(2,4,6,8),	(1,9,11,13),	(2,10,12,14),	(2,6,7,9),	(1,4,7,12),
(2,3,8,11)	(1,3,6,14),	(2,4,5,13),	(1,4,10,13),	(2,3,9,14),	(1,6,9,12)
(2,5,10,11),	(1,8,11,14),	(2,7,12,13),	(5,8,9,13),	(6,7,10,14),	(4,7,9,11),
(3,8,10,12),	(3,6,16,13),	(4,5,12,14),	(5,7,11,14),	(6,8,12,13),	(3,7,10,13),
(4,8,9,14),	(3, 5,9,12),	(4,6,10,11)			

Corollary 2.1: If the incidence structure S is written in the form as

S =

N	N	N	N	N	N	N	N	0	0	0	0	0	0
N	N	J-N	0	0	J-N	0	0	0	N	N	0	N	N
N	N	0	J-N	0	0	J-N	0	N	0	J-N	N	0	J-N
N	N	0	0	J-N	0	0	J-N	J-N	J-N	0	J-N	J-N	0
0	0	0	N	J-N	N	0	J-N	N	0	N	0	N	J-N
0	0	N	0	J-N	N	J-N	0	J-N	N	0	N	0	N
0	0	N	J-N	0	0	N	J-N	0	N	J-N	J-N	N	0

...(2.6)

Then S in the incidence matrix of a regular GD design with the parameters

$$v^* = 7v, b^* = 14b, r^* = 8\Omega, k^* = 4k, \lambda_1^* = 8\lambda, \lambda_2^* = 8r \; ; m = 7, n = v.$$

Remark 2.2:

If in S given in (2.6) take N, the incidence matrix of a BIBD (2, 2, 1, 1, 0), we get regular GD design with the same parameters given in (2.6). But this solution is new non-isomorphic to both solutions reported in Clatworthy (1973) and that in (2.5) with respect to distributions of block intersections numbers. The blocks for new non-isomorphic solution are written as follows:

(1,3,5,7), (2,4,6,8), (1,3,5,7), (2,4,6,8), (1,4,11,13), (2,3,12,14),

(1,6,9,14), (2,5,10,13), (1,8,10,12), (2,7,9,11), (1,4,9,11), (2,3,10,12),

(1,6,12,143, (2,5,11,14), (1,8,10,14), (2,7,9,13), (5,8,9,12), (6,7,10,11),

(3,8,11,13), (4,7,12,14), (3,6,9,14), (4,5,10,13), (5,8,11,14), (6,7,12,13),

(3,8,9,13), (4,7,10,14), (3,6,10,11), (4,5,9,12).

The method described in theorem 2.1, 2.2 and 2.3 are useful for the combinatorial constructions of regular GD designs, but they may produce deigns with relatively large parameter values. In the range r, k ≤ 10 of much practical value, some examples are here taken up.

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