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INVOLUTE CURVES OF BIHARMONIC REEB CURVES 3-DIMENSIONAL KENMOTSU MANIFOLD

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Abstract

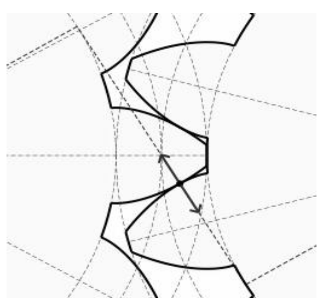
In this paper, we study involute curves of biharmonic Reeb curves in 3-dimensional Kenmotsu manifold.

Keywords: Kenmotsu manifold, biharmonic curve, Reeb vector field.

Mathematics Subject Classifications: 53C41, 53A10.

1. Introduction

The involute has some properties that makes it extremely important to the gear industry: If two intermeshed gears have teeth with the profile-shape of involutes (rather than, for example, a "classic" triangular shape), they form an involute gear system.



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A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\tau(\phi)|^2 dv_h,$$

where $\tau(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler--Lagrange equation of the bienergy is given by $\tau_2(\phi) = 0$. Here the section $\tau_2(\phi)$ is defined by

$$\tau_2(\phi) = -\Delta_{\phi}\tau(\phi) + \mathrm{tr}R(\tau(\phi), d\phi)d\phi, \quad (1.1)$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold with 1-form η, the associated vector field ξ, $(1,1)$-tensor field ϕ and the associated Riemannian metric g. It is well known that [2]

$$\phi\xi = 0, \eta(\xi) = 1, \eta(\phi X) = 0, \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, Y on M. Moreover,

$$(\nabla_X \phi)Y = -\eta(Y)\phi(X) - g(X, \phi Y)\xi, \quad X, Y \in \chi(M), \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called an Kenmotsu manifold [2].

3. Biharmonic Reeb Curves in the 3-Dimensional Kenmotsu Manifold

Let γ be a non geodesic curve on the 3-dimensional Kenmotsu manifold parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the 3-dimensional Kenmotsu manifold along γ defined as follows:

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$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}, \qquad \nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad \nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N}, \tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$g(\mathbf{T},\mathbf{T}) = 1, g(\mathbf{N},\mathbf{N}) = 1, g(\mathbf{B},\mathbf{B}) = 1, \qquad g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \tag{3.2}$$

Theorem 3.1. ([10]) *Let* (M,ϕ,ξ,η,g) *be an 3-dimensional Kenmotsu manifold and unit vector field* X *orthogonal to the Reeb vector field* ξ. *Then,*

$$R(\xi,X)\xi = X, \tag{3.3}$$

$$R(X,\xi)X = \xi. \tag{3.4}$$

Theorem 3.2. ([10]) γ *is a non geodesic biharmonic Reeb curve which are either tangent or normal to the Reeb vector field 3-dimensional Kenmotsu manifold if and only if*

$$\kappa = \text{constant} \neq 0, \qquad \kappa^2 + \tau^2 = 1, \qquad \tau = \text{constant}. \tag{3.5}$$

To determine γ we need the following result.

Corollary 3.3. *If* γ *is a non geodesic biharmonic Reeb curve which are either tangent or normal to the Reeb vector field 3-dimensional Kenmotsu manifold, then* γ *is a helix.*

Proof. From the above Theorem it can be easily seen that γ is a helix.

We consider the special 3-dimensional manifold

$$\mathrm{K} = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) \neq (0,0,0)\},$$

where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1 = z\frac{\partial}{\partial x}, \mathbf{e}_2 = z\frac{\partial}{\partial y}, \mathbf{e}_3 = -z\frac{\partial}{\partial z} \tag{3.6}$$

are linearly independent at each point of K. Let g be the Riemannian metric defined by

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = g(\mathbf{e}_3,\mathbf{e}_3) = 1,$$

$$g(\mathbf{e}_1,\mathbf{e}_2)=g(\mathbf{e}_2,\mathbf{e}_3)=g(\mathbf{e}_1,\mathbf{e}_3)=0. \tag{3.7}$$

The characterising properties of $\chi(\mathrm{K})$ are the following commutation relations:

$$[\mathbf{e}_1,\mathbf{e}_2]=0,[\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_1,[\mathbf{e}_2,\mathbf{e}_3]=\mathbf{e}_2. \tag{3.8}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\mathbf{e}_3)\text{foranyZ}\in\chi(M)$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1)=-\mathbf{e}_2,\phi(\mathbf{e}_2)=\mathbf{e}_1,\phi(\mathbf{e}_3)=0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3)=1, \tag{3.9}$$

$$\phi^2(Z)=-Z+\eta(Z)\mathbf{e}_3, \tag{3.10}$$

$$g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W), \tag{3.11}$$

for any $Z,W\in\chi(\mathrm{K})$. Thus for $\mathbf{e}_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on K.

Now, we consider biharmonicity of curves in the special three-dimensional Kenmotsu manifold K.

Theorem 3.4. ([10]) *Let $\gamma:I\to\mathrm{K}$ be a non geodesic unit speed biharmonic Reeb curve which are either tangent or normal to the Reeb vector field 3-dimensional Kenmotsu manifold K. Then, the parametric equations of γ are*

$$x(s)=\frac{C_1\sin^5\varphi}{\kappa^2+\sin^4\varphi\cos^2\varphi}e^{-\cos\varphi s}(\frac{\kappa}{\sin^2\varphi}\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)$$
$$+\cos\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma))+C_2,$$

$$y(s)=\frac{C_1\sin^5\varphi}{\kappa^2+\sin^4\varphi\cos^2\varphi}e^{-\cos\varphi s}(-\cos\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma) \tag{3.12}$$
$$+\frac{\kappa}{\sin^2\varphi}\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma))+C_3,$$

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$$z(s) = C_1 e^{-\cos\varphi s},$$

where $C,\ C_1,\ C_2,\ C_3$ *are constants of integration*.

4. Involute Curves of Biharmonic Reeb Curves in the 3-Dimensional Kenmotsu Manifold $\mathcal{K}$

Definition 4.1. *Let unit speed curve* $\gamma : I \to \mathcal{K}$ *and the curve* $\beta : I \to \mathcal{K}$ *be given. For* $\forall s \in I$, *then the curve* β *is called the involute of the curve* γ, *if the tangent at the point* $\gamma(s)$ *to the curve* γ *passes through the tangent at the point* $\beta(s)$ *to the curve* β *and*

$$g\left(\mathbf{T}^*(s), \mathbf{T}(s)\right) = 0. \tag{4.1}$$

Let the Frenet-Serret frames of the curves γ and β be $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}^*\}$, respectively.

Theorem 4.2. *Let* $\gamma : I \to \mathcal{K}$ *be a non geodesic unit speed biharmonic Reeb curve which are either tangent or normal to the Reeb vector field 3-dimensional Kenmotsu manifold* $\mathcal{K}$ *and the curve* β *be involute of the the curve* γ *and let* ρ *be a constant real number. Then, the parametric equation of involute curve* β *are*

$$\tilde{x}(s) = \frac{C_1 \sin^5\varphi}{\kappa^2 + \sin^4\varphi\cos^2\varphi} e^{-\cos\varphi s}\left(\frac{\kappa}{\sin^2\varphi}\cos\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right)\right.$$
$$\left. + \cos\varphi\sin\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right)\right) + (\rho - s)C_1 e^{-\cos\varphi s}\sin\varphi\sin\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right) + C_2, \tag{4.2}$$

$$\tilde{y}(s) = \frac{C_1 \sin^5\varphi}{\kappa^2 + \sin^4\varphi\cos^2\varphi} e^{-\cos\varphi s}\left(-\cos\varphi\cos\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right)\right.$$
$$\left. + \frac{\kappa}{\sin^2\varphi}\sin\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right)\right) + (\rho - s)C_1 e^{-\cos\varphi s}\sin\varphi\cos\left(\frac{\kappa}{\sin^2\varphi}s + \sigma\right) + C_3,$$

$$\tilde{z}(s) = C_1 e^{-\cos\varphi s} - (\rho - s)C_1 e^{-\cos\varphi s}\cos\varphi,$$

where $C,\ C_1,\ C_2,\ C_3$ *are constants of integration*.

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Proof. The curve $\beta(s)$ may be given as

$$\beta(s) = \gamma(s) + u(s)\mathbf{T}(s). \tag{4.3}$$

Since γ is biharmonic, γ is a helix. So, without loss of generality, we take the axis of γ is parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{T}, \mathbf{e}_3) = T_3 = \cos\varphi, \tag{4.4}$$

where φ is constant angle.

If we take the derivative (4.3), then we have

$$\beta'(s) = (1 + u'(s))\mathbf{T}(s) + u(s)\kappa(s)\mathbf{N}(s).$$

Since the curve β is involute of the curve γ, $g(\mathbf{T}^*(s), \mathbf{T}(s)) = 0.$ Then, we get

$$1 + u'(s) = 0 \text{or} u(s) = \rho - s, \tag{4.5}$$

where ρ is constant of integration.

Substituting (4.5) into (4.3), we get

$$\beta(s) = \gamma(s) + (\rho - s)\mathbf{T}(s). \tag{4.6}$$

On the other hand, from Theorem 3.3 we obtain

$$\mathbf{T} = \sin\varphi \sin\left(\frac{\kappa}{\sin^2\varphi} s + \sigma\right)\mathbf{e}_1 + \sin\varphi \cos\left(\frac{\kappa}{\sin^2\varphi} s + \sigma\right)\mathbf{e}_2 + \cos\varphi \mathbf{e}_3. \tag{4.7}$$

Using (3.8) in (4.7), we obtai

$$\mathbf{T} = (C_1 e^{-\cos\varphi s} \sin\varphi \sin\left(\frac{\kappa}{\sin^2\varphi} s + \sigma\right), C_1 e^{-\cos\varphi s} \sin\varphi \cos\left(\frac{\kappa}{\sin^2\varphi} s + \sigma\right), -C_1 e^{-\cos\varphi s} \cos\varphi). \tag{4.8}$$

Then from (4.8) we find the equalities (4.2). This completes the proof.

From (4.2) we can give the following result.

Corollary 4.3. *Let* $\gamma : I \to \mathrm{K}$ *be a non geodesic unit speed biharmonic Reeb curve which are either tangent or normal to the the Reeb vector field 3-dimensional Kenmotsu manifold* K *and the curve* β *be involute of the the curve* γ *and let* ρ *be a*

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constant real number. Then, the parametric equation of involute curve β *in terms of* (3.7) *are*

$$\tilde{x}(s)=\frac{C_1\sin^5\varphi}{1-\tau^2+\sin^4\varphi\cos^2\varphi}e^{-\cos\varphi s}\Big(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}\cos\Big(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma\Big)$$

$$+\cos\varphi\sin\Big(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma\Big)\Big)+(\rho-s)C_1e^{-\cos\varphi s}\sin\varphi\sin\Big(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma\Big)+C_2,$$

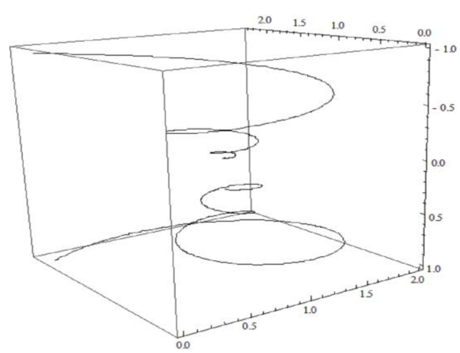
$$\tilde{y}(s)=\frac{C_1\sin^5\varphi}{1-\tau^2+\sin^4\varphi\cos^2\varphi}e^{-\cos\varphi s}\Big(-\cos\varphi\cos\Big(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma\Big)$$

$$+\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}\sin\Big(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma\Big)\Big)+(\rho-s)C_1e^{-\cos\varphi s}\sin\varphi\cos\Big(\frac{\sqrt{1-\tau^2}}{\sin^2\varphi}s+\sigma\Big)+C_3,$$

$$\tilde{z}(s)=C_1e^{-\cos\varphi s}-(\rho-s)C_1e^{-\cos\varphi s}\cos\varphi,$$

where $C,\ C_1,\ C_2,\ C_3$ *are constants of integration*.

We use Mathematica both involute curve and biharmonic curve, we have:



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