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A NOTE ON THE NAMIAS IDENTITY FOR BERNOULLI NUMBERS

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Abstract

A Namias identity for Bernoulli numbers B_r, is decomposed into two relations which together contain the same information as that identity. Then it is proved that one of these relationships can be demonstrated by elementary means, and the another simplifies a known expression for the Euler numbers in terms of B_r.

Key words: Namias identity, Bernoulli and Euler numbers.

AMS Subject Classification: 11 B65; 11 B68

1. Introduction

Namias [4,6] obtained the following identity for Bernoulli numbers [1-3,5]:

$$B_n = \frac{1}{2(1-2^n)} \sum_{r=0}^{n-1} 2^r \binom{n}{r} B_r \quad , \qquad n = 1, 2, \cdots \tag{1}$$

which for n even and odd leads to two relations which together amount to (1):

$$B_{2n} = \frac{1}{2(1-2^{2n})} \left[-1 + \sum_{k=1}^{n-1} \binom{2k}{2-2} \binom{2n}{2k} B_{2k} \right] \quad , \quad n = 1, 2, \cdots \tag{2}$$

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$$\sum_{k=0}^{N} 2^{2k} \binom{2N+1}{2k} B_{2k} = 2N+1 \quad , \quad N = 0\,,1\,,\cdots \tag{3}$$

with the convention that $\sum$ in (2) is zero for $n = 1$.

The Section 2 shows that (2) can be derived by elementary expressions that define the Bernoulli numbers, avoiding the use of the Gamma function as in the publication of Namias. The Section 3 makes a reformulation of (3), which simplifies a relationship between Euler numbers and B_r.

2. Elementary Proof of (2)

In the identity:

$$\frac{x}{2}\cot\left(\frac{x}{2}\right) = \frac{\frac{x}{2}}{\operatorname{sen}\left(\frac{x}{2}\right)}\cos\left(\frac{x}{2}\right),$$

the expressions are replaced by their Taylor expansions [1,5]:

$$\frac{x}{2}\cot\left(\frac{x}{2}\right) = 1 - B_2\frac{x^2}{2!} + B_4\frac{x^4}{4!} - B_6\frac{x^6}{6!} + \cdots,$$

$$\frac{\frac{x}{2}}{\operatorname{sen}\left(\frac{x}{2}\right)} = 1 + (2^2-2)B_2\frac{x^2}{2!\,2^2} - (2^4-2)B_4\frac{x^4}{4!\,2^4} + \cdots, \tag{4}$$

$$\cos\left(\frac{x}{2}\right) = 1 - \frac{x^2}{2!\,2^2} + \frac{x^4}{4!\,2^4} - \frac{x^6}{6!\,2^6} + \cdots$$

for (remember that $B_0 = 1$ and $B_1 = -\frac{1}{2}$):

$$B_2 = \frac{1}{2(1-2^2)}[-1] = \frac{1}{6} \quad ,$$

$$B_4 = \frac{1}{2(1-2^4)}\left[-1 + (2^2-2)\binom{4}{2}B_2\right] = -\frac{1}{30} \quad ,$$

$$B_6 = \frac{1}{2(1-2^6)}\left[-1 + (2^2-2)\binom{6}{2}B_2 + (2^4-2)\binom{6}{4}B_4\right] = \frac{1}{42} \quad , \quad etc.,$$

from it is immediate the expression (2). We emphasize that this simple demonstration used only the basic relations (4) that define the Bernoulli numbers, without the presence of the Gamma function employed by Namias [6].

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3. An Application of (3) to the Euler Numbers

It is easy to see that the result (3) accepts the reformulation:

$$\sum_{k=1}^{N} \frac{2^{2k}}{2k} \binom{2N}{2k-1} B_{2k} = \frac{2N}{2N+1} \quad , \quad N = 0, 1, 2, \cdots \tag{5}$$

with the convention that this $\sum$ is zero for $N = 0$. The Euler numbers are given by the expression [6]:

$$E_n = (-1)^n \left[1 + \sum_{k=1}^{n} \frac{2^{2k} - 4^{2k}}{2k} \binom{2n}{2k-1} B_{2k} \right] \quad , \quad n = 1, 2, \cdots \tag{6}$$

which can be simplified under (5):

$$E_n = (-1)^n \left[\frac{4n+1}{2n+1} - \sum_{k=1}^{n} \frac{4^{2k}}{2k} \binom{2n}{2k-1} B_{2k} \right] \quad , \quad n = 1, 2, \cdots \tag{7}$$

the values obtained $E_1 = 1$, $E_2 = 5$, $E_3 = 61, \cdots$, are entirely consistent with the Taylor expansion [1]:

$$\sec\left(\frac{x}{2}\right) = 1 + E_1 \frac{x^2}{2!\,2^2} + E_2 \frac{x^4}{4!\,2^4} + E_3 \frac{x^6}{6!\,2^6} + \cdots \tag{8}$$

In addition, if into the identity:

$$\frac{\frac{x}{2}}{\text{sen}\left(\frac{x}{2}\right)} = \frac{x}{2} \cot\left(\frac{x}{2}\right) \sec\left(\frac{x}{2}\right),$$

are employed the expansions (4) and (8) then a relationship between the numbers of Bernoulli and Euler, is obtained:

$$B_{2n} = \frac{(-1)^n}{2(1-2^{2n})} \sum_{k=0}^{n-1} (-1)^k \, 2^{2k} \binom{2n}{2k} E_{n-k} B_{2k} \quad , \quad n = 1, 2, \cdots \tag{9}$$

which it is difficult to find in the literature explicitly.

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