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A New Transmuted Lifetime Distribution: Statistical Properties and Application to Survival Data

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Abstract: **In statistical literature, several methods are available to generate a new probability distribution by introducing new parameter to any existing standard distribution. The quadratic rank transmutation map method is one of these and received considerable attention in the literature. Here, we also proposed a new probability distribution using this method when DUSE(θ)-distribution is chosen as baseline distribution. The proposed distribution is called transmuted DUSE(θ)-distribution, which is seems to be more flexible as compared to the baseline distribution. Different statistical properties such as moments, quantile function, survival function, hazard function and order statistics have been derived. Also, the method of maximum likelihood and method of maximum product spacing are used to estimate the unknown parameters of the introduced probability distribution. Simulation study is being carried out to know the long-run behavior of the distribution. Finally, a real data set has been utilized to show the applicability of the proposed distribution.**

Index Terms: **Maximum Likelihood Estimator, Maximum Product Spacing Estimator, QRTM, DUS-transformation, DUSE(θ)-distribution.**

I. Introduction

In statistical literature, a large number of distributions are available to analyze the characteristics of the lifetime data. The application of statistics particularly that of statistical modeling in our day to day life, is so vast that there is merely any field where statistics can't be used. To predict some future event with higher accuracy, the statistician must have to care regarding selection of model, which may achieve in term of flexibility and or parsimony of the selected models.

In survival analysis, a number of models are available in literature to study the characteristics associated with the lifetime data. Initially, exponential distribution was extensively used due to its memory less property and analytical tractability. Although the use of one-parameter exponential distribution has been restricted due to its constant failure rate and seems to be inappropriate in real-life situations where associated hazard rate is not constant. To accommodate the situation of non-constant hazard rate, researchers attempted, again and again, to develop new lifetime distributions so that these become more flexible to analyze such type of failure rate behavior. In this context, gamma and Weibull distribution have been introduced and found more suitable for the data having monotone hazard rate, e.g. See (Mudholkar et al. (1993), Gupta et al. (1998), Nadarajh et al.(2011) etc), unimodal and bathtub shape of hazard rate of TGIED discussed by Okorie, I. E., & Akpanta, A. C. (2019). But, both of these models are applicable only for monotone failure rate and found less advantageous for those real life data which shows the behavior of non-monotone failure rate. In order to get much flexible distribution, several generalization techniques has been introduced. To incorporate such flexibility in the model, several generalizations or a new class of distributions based on specified baseline distributions have been proposed in the literature. Recently, Kumar et al. (2015), advocated to transform any available baseline distribution, so that the new distribution thus obtained is parsimonious in parameters as the transformation do not incorporate any additional parameter, rather it adds flexibility in terms of wide variety of hazard rate function. They have introduced DUS transformation to obtain a new distribution.

If $H(x)$ be the cdf of a baseline distribution, then DUS transformation that provides cdf of a new distribution, say $F(x)$ as given below:

$$F(x) = \frac{e^{H(x)}-1}{e-1} \quad \dots \quad (1)$$

Shaw and Buckley (2007) introduced $QRTM$ to generalize any available distribution. It provides cdf $G(x)$ of a new distribution as follows,

$$G(x) = (1+\lambda)F(x) - \lambda(F(x))^2 \quad ; \ |\lambda| < 1 \ \dots \quad (2)$$

Here, we have considered $QRTM$ to get a new lifetime distribution and the considered baseline distribution is

$DUS_E\ (\theta)$-distribution. For detailed discussion of $DUS_E\ (\theta)$-distribution, readers may refer to Kumar et al. (2015). The pdf and cdf of $DUS_E\ (\theta)$-distribution are respectively, given by,

$$f(x;\theta) = \left(\frac{\theta e^{-\theta x}\, e^{1-e^{-\theta x}}}{e-1}\right) \quad \ldots \quad (3)$$

$$F(x;\theta) = \left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right) \quad ; x>0, \theta>0 \quad \ldots \quad (4)$$

Where, θ is the scale parameter. Now, using (2) and $DUS_E\ (\theta)$-distribution having cdf $F(x)$ as baseline distribution, the cdf of the new distribution is given by,

$$G(x) = (1+\lambda)\left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right) - \lambda\left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right)^2$$
$$= \left(1+\frac{\lambda e(1-e^{-e^{-\theta x}})}{e-1}\right)\left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right) \quad \ldots (5)$$

and the pdf corresponding to it, is given by,

$$g(x) = (1+\lambda)\left(\frac{\theta e^{-\theta x}\, e^{1-e^{-\theta x}}}{e-1}\right) - 2\lambda\left(\frac{\theta e^{-\theta x}\, e^{1-e^{-\theta x}}}{e-1}\right)\left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right)$$
$$= \left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta x}})}{e-1}\right)\left(\frac{\theta e^{-\theta x}\, e^{1-e^{-\theta x}}}{e-1}\right) \quad \ldots (6)$$

Several authors have used QRTM to get new lifetime distribution and new distribution, thus obtained is flexible in terms of applicability and different shapes of hazard rate function. For detailed discussions readers may refers to Ashour and Eltehiwi (2013), Merovci et al. (2013), Ibrahim Elbatal (2013) and Okorie, I. E. and Akpanta, A. C. (2019) etc. For frequently used purpose, we will call new distribution having pdf (5) as Transmuted $DUS_E(\theta)$ −distribution with parameter θ and will be abbreviated as $TD_E(\theta)$-distribution. The rest of the paper is organized as follows; some of the statistical properties have been discussed in section (2). Method of Estimation has been used in section (4), real data illustration; simulation study and conclusions are carried out in sections (5), (6) and (7) respectively.

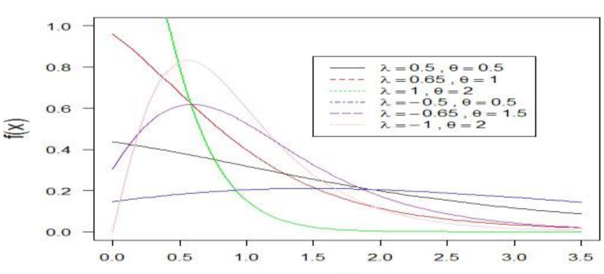


Figure1: Plots of pdf for various choice of parameters θ and λ.

II. Statistical Properties

Some important statistical properties of $TD_E(\theta)$-distribution such as survival function (SF), hazard rate (HR) function, reverse-hazard rate (RHR) function, Central Moments, mean, median, mode, variance, sample generation, skewness and kurtosis have been discussed in this section.

A. *Survival Function*

The survival function is a function that gives the probability that a patient, device, or other object of interest will survive beyond any given specified time (or age). The survival function beyond a certain point of time say x, is denoted by $S(x)$ and is obtained as follows,

$$S(x) = \left(1-\frac{e^{1-e^{-\theta x}}}{e-1}-\frac{\lambda e^{2-e^{-\theta x}}}{(e-1)^2}+\frac{\lambda e^{2-2e^{-\theta x}}}{(e-1)^2}+\frac{1}{e-1}+\frac{\lambda e}{(e-1)^2}+\frac{\lambda e^{1-e^{-\theta x}}}{(e-1)^2}\right) \quad \ldots (7)$$

The plots of survival function S(x) for different pair of values of (λ,θ) are shown in Figure (2).

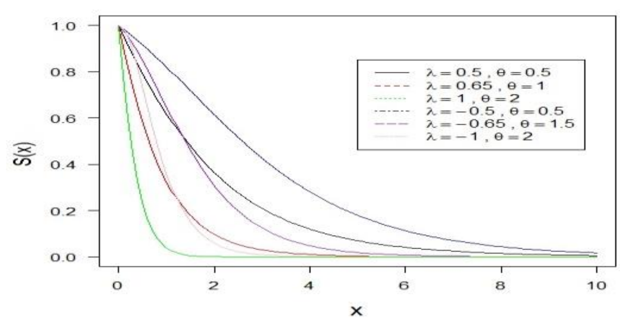


Figure 2: Plots of Survival function for various choice of parameters θ and λ.

B. *Hazard Rate Function*

The hazard rate function h(x), which is obtained as the limit of the rate at which failure occurs in a certain time interval $(x,\ x+\Delta x)$ when $\Delta x \to 0$, is obtained for $TD_E(\lambda,\theta)$ -distribution, as follows,

$$h(x) = \frac{\left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta x}})}{e-1}\right)\left(\frac{\theta e^{-\theta x}\, e^{1-e^{-\theta x}}}{e-1}\right)}{\left(\frac{e}{e-1}-\frac{e^{1-e^{-\theta x}}}{e-1}-\frac{\lambda e^{2-e^{-\theta x}}}{(e-1)^2}+\frac{\lambda e^{2-2e^{-\theta x}}}{(e-1)^2}+\frac{\lambda e}{(e-1)^2}+\frac{\lambda e^{1-e^{-\theta x}}}{(e-1)^2}\right)} \quad \ldots (8)$$

Figure 3: Plots of Hazard Rate function for various choice of parameters θ and λ.

C. *Reverse Hazard Rate Function*

The reverse hazard rate function of $TD_E(\theta)$ -distribution is obtained as follows,

$$r(x) = \frac{\left(e-1+\lambda(e+1-2e^{1-e^{-\theta x}})\right)\theta e^{-\theta x}\, e^{1-e^{-\theta x}}}{\left(e-1+\lambda(e-e^{1-e^{-\theta x}})\right)(e^{1-e^{-\theta x}}-1)} \quad \ldots (9)$$

D. Moments

On the basis of the moments, we identify the form and nature of the distribution. With the help of Moments, we obtain the bulginess, flatness and peakedness of the concerned distribution.

The r[th] moment about origin viz. $\mu_r^{'}$ for TD$_E(\theta)$ -distribution is obtained as follows,

$$\mu_r'=E(X^r) = \int_0^\infty X^r\, g(x)\, dx$$
$$=\left[\left\{1+\frac{\lambda(e+1)}{e-1}\right\}\left\{\frac{\theta}{e-1}\right\}\right]\int_0^\infty X^r\, e^{-\theta x}\, e^{1-e^{-\theta x}}\, dx - \frac{2\lambda\theta}{(e-1)^2}\int_0^\infty X^r\, e^{-\theta x}\, e^{2-2e^{-\theta x}}\, dx$$

On simplification, we get

$$\mu_r'=\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{2^{r-1/2}\Gamma(r+1)/2}{(e-1)\theta^r}\right)-\frac{\lambda}{(e-1)^2}\left(\frac{\Gamma(r+1)/2}{\theta^{r/2}}+\frac{\Gamma(r+1)/2}{\theta^r}\right) \quad ; r=1,2,3,4$$

Now, on putting $r=1,2,3,4$, we get

$$\mu_1'=\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}$$
$$\mu_2'=\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{\sqrt{\pi}}{\sqrt{2(e-1)\theta^2}}\right)-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}-\frac{\lambda}{\theta^2\,(e-1)^2}$$
$$\mu_3'=\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{2}{(e-1)\theta^3}\right)-\frac{\lambda}{(e-1)\,\theta^{3/2}}-\frac{3\lambda\sqrt{\pi}}{4\theta^3}$$
$$\mu_4'=\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{3\sqrt{\pi}}{\sqrt{2}\,(e-1)\,\theta^4}\right)-\frac{3\lambda\sqrt{\pi}}{4(e-1)^2\,\theta^2}-\frac{2\lambda}{\theta^4(e-1)^2} \quad .. (10)$$

And the variance is-

$$\mu_2=\mu_2'-(\mu_1')^2$$

$$\mu_2=\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{\sqrt{\pi}}{\sqrt{2(e-1)\theta^2}}\right)-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}-\frac{\lambda}{\theta^2\,(e-1)^2}-\left[\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}\right]^2 \quad \ldots (11)$$

$$\mu_3=\mu_3'-3\mu_2'\mu_1'+2(\mu_1')^3$$

$$\mu_3=\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{2}{(e-1)\theta^3}\right)-\frac{\lambda}{(e-1)\,\theta^{3/2}}-\frac{3\lambda\sqrt{\pi}}{4\theta^3}$$
$$-3\left(\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{\sqrt{\pi}}{\sqrt{2(e-1)\theta^2}}\right)-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}-\frac{\lambda}{\theta^2\,(e-1)^2}\right)\left(\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}\right)$$
$$+2\left[\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}\right]^3 \quad \ldots (12)$$

$$\mu_4=\mu_4'-4\mu_3'\,\mu_1'+6\mu_2'\,(\mu_1')^2-3(\mu_1')^4$$

$$\mu_4=\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{3\sqrt{\pi}}{\sqrt{2}\,(e-1)\,\theta^4}\right)-\frac{3\lambda\sqrt{\pi}}{4(e-1)^2\,\theta^2}-\frac{2\lambda}{\theta^4(e-1)^2}$$
$$-4\left(\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{2}{(e-1)\theta^3}\right)-\frac{\lambda}{(e-1)\,\theta^{3/2}}-\frac{3\lambda\sqrt{\pi}}{4\theta^3}\right)\left(\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}\right)$$
$$+6\left(\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{\sqrt{\pi}}{\sqrt{2(e-1)\theta^2}}\right)-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}-\frac{\lambda}{\theta^2\,(e-1)^2}\right)\left[\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}\right]^2$$
$$-3\left[\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}\right]^4 \ldots (13)$$

After evaluating these we can obtain the value of

$$\beta_1=\frac{\mu_3{}^2}{\mu_2{}^3};\ \beta_2=\frac{\mu_4}{\mu_2{}^2}$$
$$\gamma_1=\sqrt{\beta_1};\ \gamma_2=\beta_2-3 \qquad \ldots (14)$$

Finally, $\forall\ |\lambda|<1$ and $\theta>0$ distribution will be positively skewed & leptokurtic and platykurtic at $\lambda=\pm 0{:}5$, $\theta=3.5$.

E. Coefficient of Variation

The coefficient of variation (C.V.) is the measure of relative variability and is the ratio of standard deviation to mean of the considered distribution. So, using equation (10) and (11) the required expression of coefficient of variation for the TD$_E(\lambda,\theta)$ -distribution is given by-

$$C.V.=\frac{\sqrt{\left[\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{\sqrt{\pi}}{\sqrt{2(e-1)\theta^2}}\right)-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}-\frac{\lambda}{\theta^2\,(e-1)^2}-\left[\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}\right]^2\right]}}{\left[\left(1+\frac{\lambda(e+1)}{e-1}\right)\left(\frac{1}{\theta(e-1)}\right)-\frac{\lambda}{\sqrt{\theta}(e-1)^2}-\frac{\lambda\sqrt{\pi}}{2\theta(e-1)^2}\right]} \quad \ldots (15)$$

Table 1: Descriptive measures of the proposed model for different variation of parameters

λ,θ	Mean	Variance	Skewness	Kurtosis	CV
0.5,0.5	1.8819	1.5498	0.7512	3.5093	0.6615
0.5,1.0	0.8915	0.4033	0.8760	3.5551	0.7123
0.5,1.5	0.5690	0.1754	1.0409	3.7710	0.7360
0.5,2.0	0.3173	0.0551	1.0791	3.7811	0.7394
0.5,3.5	0.2127	0.0220	0.2863	0.4768	0.6976
-0.5,0.5	0.4436	0.5410	3.1802	5.1923	1.6580
-0.5,1.0	0.2713	0.1857	2.5467	4.5942	1.5884
-0.5,1.5	0.2062	0.1060	2.6425	5.0075	1.5792
-0.5,2.0	0.1478	0.0556	3.1298	6.0086	1.5952
-0.5,3.5	0.2127	0.0220	0.2863	0.4768	0.6976

For varying λ and fixed value of $\theta=0.5$

λ	Mean	Variance	Skewness	Kurtosis	CV
0.2	1.4504	1.6816	0.8817	3.5543	0.8940
0.	1.738	1.6351	0.7034	3.4540	0.735

4	1				7
0.6	2.0258	1.4232	0.9820	3.5963	0.5889
0.8	2.3134	1.0457	3.9308	3.2617	0.4420

For varying λ and fixed value of $\theta = 2.5$

-0.2	0.1986	0.0615	1.8291	4.5284	1.2480
-0.4	0.1647	0.0581	2.5866	5.3786	1.4633
-0.6	0.1308	0.0525	3.8370	6.8414	1.7508
-0.8	0.0969	0.0445	6.0826	9.5236	2.1771

F. *Sample Generation*

Inverse cdf method is one of the popular methods for sample generation of any distribution. So, the random sample for TD$_E$(θ)-distribution can be generated by using inverse cdf transformation method is as follows.

Generate random deviates (U) from uniform distribution. Equate G(x) = U ⇒ x = G^{-1} (U); where U itself follow the uniform distribution over 0 and 1.

$$\Rightarrow \left(1+\frac{\lambda e(1-e^{-e^{-\theta x}})}{e-1}\right)\left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right)=U \qquad \dots (16)$$

After simplification, we get

$$\Rightarrow x = -\frac{1}{\theta}\ln\left[\ln\left(\frac{(e-1+\lambda(e+1))\pm(e-1)\sqrt{(1+\lambda)^2-4U\lambda}}{2e\lambda}\right)\right] \qquad \dots(17)$$

G. *Quantiles*

The quantile function of the q^{th} order of TD$_E$(θ)-distribution is the solution of the equation.

$$G(x_q) = q$$

$$\Rightarrow x_q = -\frac{1}{\theta}\ln\left[\ln\left(\frac{(e-1+\lambda(e+1))\pm(e-1)\sqrt{(1+\lambda)^2-4q\lambda}}{2e\lambda}\right)\right] \qquad \dots (18)$$

H. *Median*

For any distribution median is the value which divides total probability area into two equal parts. So, the median is the solution of the following

$$G(M) = \frac{1}{2} \qquad \dots (19)$$

Hence, on putting $q = 0.5$ in above equation (18), we obtain the value of median for TD$_E$(θ) distribution is given by

$$M = -\frac{1}{\theta}\ln\left[\ln\left(\frac{(e-1+\lambda(e+1))\pm(e-1)\sqrt{1+\lambda^2}}{2e\lambda}\right)\right] \qquad \dots (20)$$

I. *Shapes*

One can observe the nature of any lifetime distribution by knowing shapes of its hazard rate function and pdf. Figures (1) and (3) shows the plots of pdf and hazard rate function for different values of its parameter θ. Moreover, according to the results of Glaser (1980), we can also determine the nature of hazard rate function using some simple mathematics as discussed below:

$$\eta(t) = -\frac{f'(t)}{f(t)}$$

Here,

$$f(t) = \left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta t}})}{e-1}\right)\left(\frac{\theta e^{-\theta t}\, e^{1-e^{-\theta t}}}{e-1}\right)$$

$$\Rightarrow \ln f(t) = \ln\left(\frac{e-1+\lambda(e+1-2e^{1-e^{-\theta t}})}{e-1}\right) + \ln\theta - \theta t + (1-e^{-\theta t}) - \ln(e-1)$$

On differentiating above w. r. t. $'t'$, we get

So,

$$\eta(t) = -\frac{f'(t)}{f(t)} = \left(\frac{2\lambda\theta e^{-\theta t}\, e^{1-e^{-\theta t}}}{(e-1)+\lambda(e+1-2e^{1-e^{-\theta t}})}\right) + \theta - \theta e^{-\theta t}$$

Again on differentiating w. r. t. $'t'$, we get

$$\eta'(t) = \left(\frac{2\lambda\theta^2 e^{-\theta t}\, e^{1-e^{-\theta t}}\left(\left((e-1)+\lambda\left(e+1-2e^{1-e^{-\theta t}}\right)\right)(e^{-\theta t}-1)-2\lambda e^{-\theta t}\, e^{1-e^{-\theta t}}\right)}{\left((e-1)+\lambda\left(e+1-2e^{1-e^{-\theta t}}\right)\right)^2}\right) + \theta^2 e^{-\theta t}$$

For all values of λ and θ, we get the non-monotone failure rate pattern.

J. *Mode*

For any distribution mode is the value which have maximum probability area. So mode for TD$_E$(θ)-distribution is the value for which $g(x)$ is maximum. i.e. mode is the solution of $g'(x)$ = 0 for which $g''(x)$ < 0.

Here,

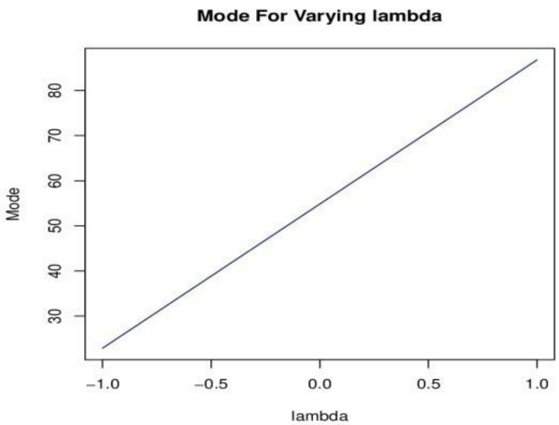
$$g(x) = \left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta x}})}{e-1}\right)\left(\frac{\theta e^{-\theta x}\, e^{1-e^{-\theta x}}}{e-1}\right)$$

$$\Rightarrow g'(x) = \left[\left\{1+\frac{\lambda(e+1)}{e-1}\right\}\left\{\frac{\theta}{e-1}\right\}\right]\left(-\theta e^{-\theta x}\, e^{1-e^{-\theta x}} + \theta e^{-2\theta x}\, e^{1-e^{-\theta x}}\right) - \frac{2\lambda\theta}{(e-1)^2}\left(-\theta e^{-\theta x}\, e^{2-2e^{-\theta x}} + 2\,\theta e^{-2\theta x}\, e^{2-2e^{-\theta x}}\right)$$

It is obvious that λ > 0, $g(x)$ is a decreasing function of x and hence $x = 0$ is the mode of this distribution, and λ < 0 mode will be at greater than 0.5 ∀ θ.

III. Order Statistics

In life testing and reliability analysis order statistics play vital role and have wide no. of applications. The importance of order statistics is arises in the theory of reliability. Let $X_{(1:n)}$; $X_{(2:n)}$;; :::; $X_{(n:n)}$;be the random sample from the TD$_E$(θ)-distribution with probability density function and cumulative distribution function shown in equation (6) & (5) respectively.



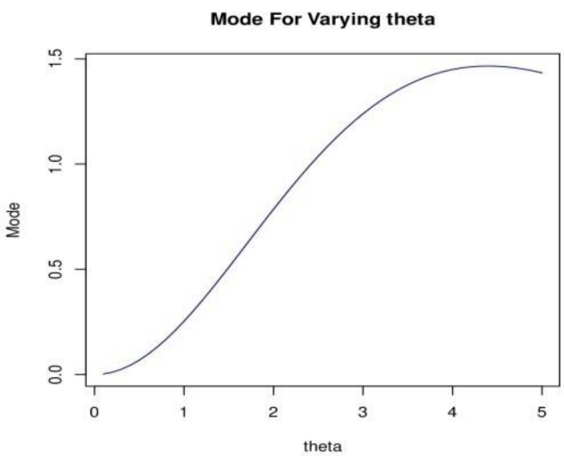


Figure 4: Mode plot for varying θ and λ

Now, let $X_{(1:n)}; X_{(2:n)};; :::; X_{(n:n)}$ be the order statistics taken from the considered random sample. In reliability studies $X_{(i:n)}$ shows the lifetime of an $(n-i+1)$ out of n systems which consists of independently and identically n components in the system. The pdf of r^{th} order statistics is given by-

$$g'(x_{r:n};\zeta) = \frac{1}{B(r,n-r+1)}\left[G(x_{(r)},\zeta)\right]^{r-1}\left[1-G(x_{(r)},\zeta)\right]^{n-r} g(x_{(r)},\zeta) \qquad ;\zeta=(\theta,\lambda)$$

A. Distribution of minimum and maximum order Statistics

In sequence of ordered sample $X_{1:n} < X_{2:n} < :::; < X_{n:n}$, $X_{1:n}$ and $X_{n:n}$ denotes the minimum and maximum order statistics. It is denoted by X_{Min} and X_{Max} respectively.

Hence,

$X_{\text{Min}} = X_{1:n} = Min\,(X_{(1:n)}; X_{(2:n)}\ ; :::; X_{(n:n)})$

$X_{\text{Max}} = X_{n:n} = Max\,(X_{(1:n)}; X_{(2:n)}\ ; :::; X_{(n:n)})$

Also, the distribution of X_{Max} and X_{Min} are obtained by simply by putting $r = n, 1$ in above equation respectively.

$$G'_{n:n}(x) = \left[\left(1+\frac{\lambda(e-e^{1-e^{-\theta x_{(n:n)}}})}{(e-1)}\right)\left(\frac{e^{1-e^{-\theta x_{(n:n)}}}-1}{(e-1)}\right)\right]^n$$

$$G'_{1:n}(x) = \left[1-\left(\frac{e}{e-1}-\frac{e^{1-e^{-\theta x_{(1:n)}}}}{(e-1)}-\frac{\lambda e^{2-e^{-\theta x_{(1:n)}}}}{(e-1)^2}+\frac{\lambda e^{2-2e^{-\theta x_{(1:n)}}}}{(e-1)^2}+\frac{\lambda e}{(e-1)^2}+\frac{\lambda e^{1-e^{-\theta x_{(1:n)}}}}{(e-1)^2}\right)^n\right]$$

$$g'_{n:n}(x) = n\left[\left(1+\frac{\lambda(e-e^{1-e^{-\theta x_{(n:n)}}})}{(e-1)}\right)\left(\frac{e^{1-e^{-\theta x_{(n:n)}}}-1}{(e-1)}\right)\right]^{n-1} \times\left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta x_{(n:n)}}})}{e-1}\right)\left(\frac{\theta e^{-\theta x_{(n:n)}}\,e^{1-e^{-\theta x_{(n:n)}}}}{e-1}\right) \quad \ldots (21)$$

$$g'_{1:n}(x) = n\left[\left(\frac{e}{e-1}-\frac{e^{1-e^{-\theta x_{(1:n)}}}}{(e-1)}-\frac{\lambda e^{2-e^{-\theta x_{(1:n)}}}}{(e-1)^2}+\frac{\lambda e^{2-2e^{-\theta x_{(1:n)}}}}{(e-1)^2}+\frac{\lambda e}{(e-1)^2}+\frac{\lambda e^{1-e^{-\theta x_{(1:n)}}}}{(e-1)^2}\right)^{n-1}\right]\times\left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta x_{(1:n)}}})}{e-1}\right)\left(\frac{\theta e^{-\theta x_{(1:n)}}\,e^{1-e^{-\theta x_{(1:n)}}}}{e-1}\right) \quad \ldots (22)$$

B. The Joint PDF Of r^{th} and s^{th} order statistics

The joint pdf of r^{th} and s^{th} order statistics of the $\text{TD}_{\text{E}}(\theta)$-distribution is given by-

$$g_{rs}(x,y) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\,[G(x)]^{r-1}\ g(x)\ [G(y)-G(x)]^{s-r-1}\ g(y)\ [1-G(y)]^{n-s}$$

$$= C_{rs}\left[\left(1+\frac{\lambda(e-e^{1-e^{-\theta x}})}{e-1}\right)\left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right)\right]^{r-1}\times\left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta x}})}{e-1}\right)\left(\frac{\theta e^{-\theta x}\,e^{1-e^{-\theta x}}}{e-1}\right)$$
$$\times\left[\left(1+\frac{\lambda\left(e-e^{1-e^{-\theta y}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta y}}-1}{e-1}\right)-\left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right)\right]^{s-r-1}$$
$$\times\left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta y}})}{e-1}\right)\left(\frac{\theta e^{-\theta y}\,e^{1-e^{-\theta y}}}{e-1}\right)$$
$$\times\left[1-\left(1+\frac{\lambda(e-e^{1-e^{-\theta y}})}{e-1}\right)\left(\frac{e^{1-e^{-\theta y}}-1}{e-1}\right)\right]^{n-s} \quad \ldots (23)$$

C. Distribution of Sample Range $W = X_{(n)} - X_{(1)}$.

We know that the probability density function of sample range $W = X_{(n)} - X_{(1)}$ is given below

$g(w) = n(n-1)\int_{-\infty}^{\infty} g(x)\,[G(x+w)] - G(x))^{n-2}\, g(x+w)\,dx$ … (24)

$$=n(n-1)\int_{-\infty}^{\infty}\left[\left(\left(1+\frac{\lambda\left(e-e^{1-e^{-\theta(x+w)}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta(x+w)}}-1}{e-1}\right) - \left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x)}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta x}}-1}{e-1}\right)\right]^{n-2} \times \left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta(x+w)}})}{e-1}\right)\left(\frac{\theta e^{-\theta(x+w)}\, e^{1-e^{-\theta(x+w)}}}{e-1}\right) \times \left(1+\frac{\lambda(e+1-2e^{1-e^{-\theta x}})}{e-1}\right)\left(\frac{\theta e^{-\theta x}\, e^{1-e^{-\theta x}}}{e-1}\right) dx$$

IV. Parameter Estimation

In this section, we consider two methods of estimation of parameter to estimate the unknown parameters namely maximum likelihood estimation method and maximum product spacing.

A. Maximum Likelihood Estimation

In this section, we discuss the method transformed by C.F.Gauss and methodology given by Prof. R.A. Fisher called as maximum likelihood estimators (MLE's) and draw the valid inferences about the $TD_E(\theta)$-distribution. Let $X_1, X_2, \ldots, X_n$ be the random sample from the population with pdf $g(x;\,\theta)$; then the likelihood function is denoted by L and the value of parameter which maximizes $L(x;\,\theta)$ will be MLE's and the procedure is as follows -

$L=\prod_{i=1}^{n} g(x_i;\theta,\lambda)$

$$= \prod_{i=1}^{n}\left(1+\frac{\lambda\left(e+1-2e^{1-e^{-\theta x_i}}\right)}{e-1}\right)\left(\frac{\theta e^{-\theta x_i}\, e^{1-e^{-\theta x_i}}}{e-1}\right)$$

$$\Rightarrow \ln L = C + \sum_{i=1}^{n}\ln\left((e-1)+\lambda\left(e+1-2e^{1-e^{-\theta x_i}}\right)\right) + n\ln\theta - \theta\sum_{i=1}^{n} x_i + \mathrm{n} - \sum_{i=1}^{n} e^{-\theta x_i}$$

Now, on differentiating partially w.r.t. parameters θ and λ respectively, we get

$$\frac{\partial \ln \mathrm{L}}{\partial\theta}=\sum_{i=1}^{n}\left(\frac{-2\lambda x_i\, e^{-\theta x_i}\, e^{1-e^{-\theta x_i}}}{(e-1)+\lambda\left(e+1-2e^{1-e^{-\theta x_i}}\right)}\right)+\frac{n}{\theta}-\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i\; e^{-\theta x_i} = 0$$

$$\frac{\partial \ln \mathrm{L}}{\partial\lambda} = \sum_{i=1}^{n}\left(\frac{e+1-2e^{1-e^{-\theta x_i}}}{(e-1)+\lambda\left(e+1-2e^{1-e^{-\theta x_i}}\right)}\right) = 0 \qquad \ldots (25)$$

These equations cannot be solved analytically. So, we use the $nlm()$ function via R-software to obtain the solution.

B. Maximum Product Spacing method of Estimation

In this section, we introduce the step by step procedure for obtaining the spacing. This method was initiated by Cheng and Amin (2011) as an alternative improvement over MLE in case of heavy-tailed distribution. The expressions of uniform spacing are as follows-

$$D_1 = F(x_1) = \left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x_1}}\right)}{(e-1)}\right)\left(\frac{e^{1-e^{-\theta x_1}}-1}{(e-1)}\right)$$

$$D_{n+1} = 1 - F(x_n) = S(x_n)$$
$$= \left(\frac{e}{e-1}-\frac{e^{1-e^{-\theta x_n}}}{(e-1)}-\frac{\lambda e^{2-e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda e^{2-2e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda e}{(e-1)^2}+\frac{\lambda e^{1-e^{-\theta x_n}}}{(e-1)^2}\right)$$

And the general term of spacing is given by

$$D_i = F(x_i) - F(x_{i-1})$$
$$= \left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x_i}}\right)}{(e-1)}\right)\left(\frac{e^{1-e^{-\theta x_i}}-1}{(e-1)}\right) - \left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x_{i-1}}}\right)}{(e-1)}\right)\left(\frac{e^{1-e^{-\theta x_{i-1}}}-1}{(e-1)}\right) \quad ; \sum_{i=1}^{n+1} D_i = 1$$

MPS method chooses the value of θ which maximizes the product of spacing or equivalently it maximizes the geometric mean of the spacing that is

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}} \qquad \ldots (26)$$

Taking logarithm on both sides, we get

$$S = \ln G = \frac{1}{n+1}\left(\sum_{i=1}^{n+1}\ln D_i\right)$$
$$=\frac{1}{n+1}\left(\ln D_1 + \sum_{i=2}^{n}\ln D_i + \ln\ D_{n+1}\right)$$
$$=\frac{1}{n+1}\left[\ln\left(\left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x_1}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta x_1}}-1}{e-1}\right)\right)\right]$$
$$+\frac{1}{n+1}\sum_{i=2}^{n}\ln\left[\left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x_i}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta x_i}}-1}{e-1}\right)-\left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x_{i-1}}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta x_{i-1}}}-1}{e-1}\right)\right]$$
$$+\frac{1}{n+1}\ln\left(\left(\frac{e}{e-1}-\frac{e^{1-e^{-\theta x_n}}}{e-1}-\frac{\lambda e^{2-e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda e^{2-2e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda e}{(e-1)^2}+\frac{\lambda e^{1-e^{-\theta x_n}}}{(e-1)^2}\right)\right) \qquad \ldots (27)$$

Now, differentiating (27) partially w.r.t. the parameters θ and λ and taking the derivatives equal to zero, we get

$$\frac{\partial S}{\partial \theta}=\frac{1}{n+1}\left[\frac{-\lambda x_1\, e^{-\theta x_1}\, e^{1-e^{-\theta x_1}}}{e-1}+\frac{x_1\, e^{-\theta x_1}\, e^{1-e^{-\theta x_1}}}{\left(e^{1-e^{-\theta x_1}}-1\right)}\right]+\frac{1}{n+1}\sum_{i=2}^{n}\left(\frac{c_1}{c_2}\right)+\frac{1}{n+1}\left(\frac{c_3}{c_4}\right)=0 \quad \ldots (28)$$

Where,

$$c_1=\frac{x_i\, e^{-\theta x_i}\, e^{1-e^{-\theta x_i}}}{e-1}+\frac{\lambda x_i\, e^{-\theta x_i}\, e^{2-e^{-\theta x_i}}}{(e-1)^2}-\frac{\lambda x_i\, e^{-2\theta x_i}\, e^{1-e^{-\theta x_i}}}{(e-1)^2}+\frac{\lambda x_i\, e^{-\theta x_i}\, e^{1-e^{-\theta x_i}}}{(e-1)^2}-\frac{x_{i-1}\, e^{-\theta x_{i-1}}\, e^{1-e^{-\theta x_{i-1}}}}{e-1}-\frac{\lambda\, x_{i-1}\, e^{-\theta x_{i-1}}\, e^{2-e^{-\theta x_{i-1}}}}{(e-1)^2}+\frac{\lambda\, x_{i-1}\, e^{-2\theta x_{i-1}}\, e^{1-e^{-\theta x_{i-1}}}}{(e-1)^2}-\frac{\lambda\, x_{i-1}\, e^{-\theta x_{i-1}}\, e^{1-e^{-\theta x_{i-1}}}}{(e-1)^2}$$

$$c_2=\left[\left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x_i}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta x_i}}-1}{e-1}\right)-\left(1+\frac{\lambda\left(e-e^{1-e^{-\theta x_{i-1}}}\right)}{e-1}\right)\left(\frac{e^{1-e^{-\theta x_{i-1}}}-1}{e-1}\right)\right]$$

$$c_3=\frac{-x_n\, e^{-\theta x_n}\, e^{1-e^{-\theta x_n}}}{e-1}-\frac{\lambda x_n\, e^{-\theta x_n}\, e^{2-e^{-\theta x_n}}}{(e-1)^2}+\frac{2\lambda\, x_n\, e^{-\theta x_n}\, e^{2-2e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda\, x_n\, e^{-\theta x_n}\, e^{1-e^{-\theta x_n}}}{(e-1)^2}$$

$$c_4=\left(\frac{e}{e-1}-\frac{e^{1-e^{-\theta x_n}}}{e-1}-\frac{\lambda e^{2-e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda e^{2-2e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda e}{(e-1)^2}+\frac{\lambda e^{1-e^{-\theta x_n}}}{(e-1)^2}\right)$$

And

$$\frac{\partial S}{\partial \lambda}=\frac{1}{n+1}\left[\frac{e-e^{1-e^{-\theta x_1}}}{(e-1)+\lambda\left(e-e^{1-e^{-\theta x_1}}\right)}\right]+\frac{1}{n+1}\left[\frac{\frac{e}{(e-1)^2}-\frac{e^{2-e^{-\theta x_n}}}{(e-1)^2}+\frac{e^{2-2e^{-\theta x_n}}}{(e-1)^2}+\frac{e^{1-e^{-\theta x_n}}}{(e-1)^2}}{c_a}\right]$$

$$+\frac{1}{n+1}\sum_{i=2}^{n}\left[\frac{\left(\left(e-e^{1-e^{-\theta x_i}}\right)\left(e^{1-e^{-\theta x_i}}-1\right)-\lambda\left(e-e^{1-e^{-\theta x_{i-1}}}\right)\left(e^{1-e^{-\theta x_{i-1}}}-1\right)\right)}{\left(\left((e-1)+\lambda\left(e-e^{1-e^{-\theta x_i}}\right)\right)\left(e^{1-e^{-\theta x_i}}-1\right)-\left((e-1)+\lambda\left(e-e^{1-e^{-\theta x_{i-1}}}\right)\right)\left(e^{1-e^{-\theta x_{i-1}}}-1\right)\right)}\right]=0 \quad \ldots (29)$$

Where,

$$c_a=\left(\frac{e}{e-1}-\frac{e^{1-e^{-\theta x_n}}}{(e-1)}-\frac{\lambda e^{2-e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda e^{2-2e^{-\theta x_n}}}{(e-1)^2}+\frac{\lambda e}{(e-1)^2}+\frac{\lambda e^{1-e^{-\theta x_n}}}{(e-1)^2}\right) \quad \ldots (30)$$

The equation (28) and (29) can't be solved analytically for θ and λ. So, we have used some iterative method to solve them numerically.

V. Real Data Illustration

In this section, a real data of the remission times of 128 bladder cancer patients has been taken from Lee and Wang (2013). To trace the failure rate pattern of the introduced model, total time on test (TTT) plot has been plotted and identified that the considered data set exhibits the pattern of upside down bathtub shape hazard rate function (see, Figure 5). Recently, Kumar et al. (2015) have considered the same dataset and showed that $DUS_E(\theta)$-distribution fits better as compared to TIWD, TIED, TIRD and IWD distributions, in terms of AIC, BIC and K-S test statistic. The proposed distribution is compared with above mentioned distributions in terms of AIC, BIC, log-likelihood and K-S test statistic. We computed the values of MLEs of the parameters θ and λ of $TD_E(\theta)$-distribution having pdf (3) for the above dataset and found them as 0.09239901 and 0.71136328 respectively, in order to calculate the values of the above criterion. The values of AIC, BIC, log-likelihood and K-S test statistic are extracted from Kumar et al. (2015). The values have been shown in comparative Table 2. From this table, it is obtained that the values of the considered criterion are least for $TD_E(\theta)$-distribution as compared to those of TIWD, IWD, TIED, TIRD and $DUS_E(\theta)$-distribution and hence we can say that our proposed distribution fits better as compared to the other considered distributions in terms of AIC, BIC, log-likelihood and K-S test statistic value.

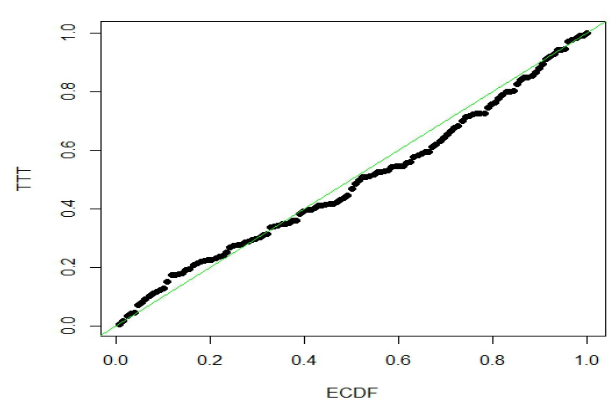


Figure 5: TTT plot for the considered real data set

Table 2: AIC, BIC, -LL and KS test value for TD$_E(\theta)$-distribution

Distributions	AIC	BIC	-LL	KS
TD$_E(\theta)$-	808.144	813.848	413.900	0.061
DUS$_E(\theta)$	834.044	836.896	415.300	0.081
TIWD	879.400	879.700	438.500	0.119
IWD	892.000	892.200	444.000	0.131
TIED	889.600	889.800	442.800	0.155
TIRD	1424.400	1424.600	710.200	0.676

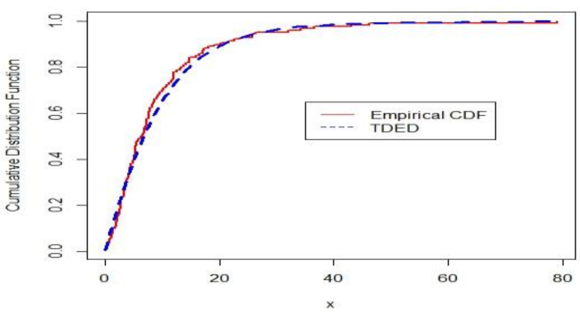


Figure 6: Plots of ECDF and cdf of fitted distribution $TD_E(\theta)$-distribution for the considered dataset.

VI. Simulation Study

In this section, the simulation study is carried out to know the performance of the considered estimators in terms of their MSEs with varying sample sizes. For this purpose, we have arbitrarily chosen $\lambda=\pm 0.5$, θ =1.0, 1.5, 2.0, 2.5 and n=10, 15, 20, 30, 50. We have calculated MLEs and MPSEs of θ and λ and repeating this process 10,000 times to obtain MSEs of MLES and MPSEs of θ and λ. The MSEs are reported in Tables 3-6. The MSE of the estimator $\hat{\theta}$ of the parameter θ based on 10,000 simulated samples is calculated by the following formula

$$MSE\ (\hat{\theta}) = \frac{1}{10{,}000} \sum_{m=1}^{10{,}000} \left(\theta - \hat{\theta}_i\right)^2$$

Where, $\hat{\theta}_i$ is the value of $\hat{\theta}$ for the i^{th} simulated sample.

It is found that the MSEs decreases as sample size increases for MLE but in case of MPS, no definite trend is found. By invariance property of MLE & MPS, we have obtained the MLEs and MPSEs and consequently their MSEs for survival function $S(x)$, hazard function $h(x)$, mean time to failure (MTTF) & median time to failure at fixed values $t = 0.5, 1.0$, by putting the estimates in (7), (8), (10) & (19) respectively. We have also found that the MLEs and MPSEs of θ and λ is least as compared to those of MLEs and MPSEs of θ and λ for $\lambda = 0.5$ and reverse trend is noted for $\lambda = -0.5$. While, in the case of estimators of survival function, hazard function, MTTF, & median time to failure, no definite trend of MSEs of their estimators is noted.

Table 3: MSE_s of MLE_s of θ and λ for fixed value of $\lambda = 0.5$.

n	θ	$\hat{\theta}_{ml}$	$\hat{\lambda}_{ml}$	$\hat{\mu}(0.5)$	$\hat{\nu}(0.5)$	$\hat{\mu}(1.0)$	$\hat{\nu}(1.0)$	$\hat{R}t_{ml}(0.5)$	$\hat{H}t_{ml}(0.5)$	$\hat{R}t_{ml}(1.0)$	$\hat{H}t_{ml}(1.0)$
10	0.5	0.04794780	0.29905370	0.71306500	0.01713739	0.69753770	0.01857460	0.00050610	0.00002671	0.00056614	0.00009698
	1.0	0.16495440	0.28272170	0.13844280	0.00358213	0.13711700	0.00291833	0.00046516	0.00609099	0.00036527	0.00745786
	1.5	0.22725190	0.22359210	0.03621249	0.00337829	0.03673636	0.00328508	0.00065126	0.01622443	0.00062300	0.01759280
	2.0	0.35478540	0.19703050	0.01307777	0.00210101	0.01247994	0.00250141	0.00050686	0.05643861	0.00020453	0.15290480
	2.5	0.51261940	0.18709120	0.00554612	0.00162322	0.00589598	0.00142671	0.00055024	0.12015070	0.00011505	0.37326660
15	0.5	0.03058519	0.23765160	0.57335190	0.01558098	0.59809880	0.01548049	0.00058026	0.00052551	0.00061549	0.00059161
	1.0	0.14085430	0.24708940	0.12409390	0.00190878	0.12251030	0.00197090	0.00041127	0.00393395	0.00041158	0.00381153
	1.5	0.24029330	0.21566170	0.03723244	0.00156163	0.03691916	0.00150618	0.00030759	0.02055323	0.00027984	0.02170607
	2.0	0.35643570	0.19343310	0.01507202	0.00101377	0.01462713	0.00110202	0.00018373	0.05967174	0.00000086	0.19010450
	2.5	0.45809270	0.16493010	0.00621069	0.00072739	0.00629927	0.00069238	0.00012773	0.11568020	0.00000263	0.33946910
20	0.5	0.02050557	0.17786600	0.42028690	0.01311605	0.48914050	0.01309638	0.00051172	0.00075804	0.00056371	0.00085514
	1.0	0.11124670	0.19485940	0.10146610	0.00127001	0.10748430	0.00147099	0.00029391	0.00261381	0.00036948	0.00226089
	1.5	0.20145590	0.18296070	0.03265207	0.00105994	0.03325476	0.00100547	0.00023815	0.01516288	0.00021375	0.01634150
	2.0	0.25913520	0.15758950	0.01213937	0.00086310	0.01180155	0.00093128	0.00021434	0.03383297	0.00000003	0.13704080
	2.5	0.33428010	0.13981480	0.00500512	0.00068128	0.00503899	0.00067509	0.00016018	0.07250832	0.00000076	0.23702620
30	0.5	0.01317016	0.13336320	0.33362290	0.01045207	0.32720650	0.01014949	0.00041988	0.00101060	0.00041017	0.00095597
	1.0	0.07009094	0.13575830	0.07284339	0.00104213	0.07362803	0.00094255	0.00026979	0.00053328	0.00025424	0.00065180
	1.5	0.15259390	0.13789470	0.02649914	0.00059294	0.02614761	0.00056445	0.00014520	0.00941346	0.00013039	0.00962993
	2.0	0.21318270	0.12762390	0.01039304	0.00053766	0.01058097	0.00048436	0.00012504	0.02553156	0.00001487	0.11754570
	2.5	0.26744290	0.10857560	0.00425337	0.00038741	0.00444790	0.00036950	0.00006173	0.05735009	0.00000727	0.19537200

50	0.5	0.00699935	0.07504531	0.19324430	0.00591320	0.18921930	0.00561520	0.00022550	0.00068811	0.00032323	0.00006277
	1.0	0.04049625	0.08067348	0.04578980	0.00057365	0.04428807	0.00060650	0.00015234	0.00008709	0.00016236	0.00005653
	1.5	0.08818562	0.08300309	0.01692097	0.00034040	0.01681790	0.00029986	0.00010006	0.00343406	0.00008353	0.00379999
	2.0	0.14493440	0.08398297	0.00791280	0.00024832	0.00751214	0.00021824	0.00005322	0.01392612	0.00003027	0.08021830
	2.5	0.16330770	0.06306164	0.00282176	0.00016443	0.00272084	0.00017776	0.00001565	0.03260641	0.00001164	0.11582670

Table 4: MSE_s of MPS_s of θ and λ for fixed value of $\lambda = 0.5$.

n	θ	$\hat{\theta}_{mp}$	$\hat{\lambda}_{mp}$	$\hat{\mu}(0.5)$	$\hat{v}(0.5)$	$\hat{\mu}(1.0)$	$\hat{v}(1.0)$	$\hat{R}t_{mp}(0.5)$	$\hat{H}t_{mp}(0.5)$	$\hat{R}t_{mp}(1.0)$	$\hat{H}t_{mp}(1.0)$
10	0.5	0.00177051	0.02998620	0.00013436	0.05273334	0.00000052	0.05576738	0.00028885	0.00074888	0.00034050	0.00100567
	1.0	0.00390870	0.02681661	0.00005885	0.01194386	0.00002341	0.01080773	0.00085840	0.00209700	0.00074640	0.00150800
	1.5	0.00000357	0.01667482	0.00342927	0.00900478	0.00305165	0.00884236	0.00235484	0.01262478	0.00231687	0.01175574
	2.0	0.00062813	0.01250220	0.00402118	0.00566560	0.00454282	0.00635872	0.00311942	0.01996912	0.00251448	0.00719433
	2.5	0.00196614	0.01159274	0.00383588	0.00418083	0.00325716	0.00384591	0.00371884	0.02702312	0.00159847	0.00085265
15	0.5	0.00120578	0.02879645	0.00174400	0.03733089	0.00263871	0.03643409	0.00035477	0.00122519	0.00035840	0.00125205
	1.0	0.00766319	0.02988656	0.00103559	0.00585311	0.00062571	0.00594769	0.00056125	0.00083761	0.00056368	0.00090447
	1.5	0.00600891	0.02355736	0.00010831	0.00396029	0.00010798	0.00388936	0.00113453	0.00254721	0.00109415	0.00216623
	2.0	0.00279944	0.01642821	0.00066817	0.00249324	0.00071169	0.00265386	0.00142753	0.00353372	0.00080359	0.00071380
	2.5	0.00001261	0.00948742	0.00136616	0.00179430	0.00125253	0.00174481	0.00160222	0.00566308	0.00056464	0.00243821
20	0.5	0.00073242	0.02323471	0.00087111	0.02891675	0.00354663	0.02765189	0.00035017	0.00136490	0.00036393	0.00139217
	1.0	0.00800606	0.02579462	0.00123919	0.00374564	0.00245085	0.00403203	0.00040579	0.00045490	0.00046896	0.00059034
	1.5	0.00923360	0.02444900	0.00003134	0.00252577	0.00009203	0.00243783	0.00079793	0.00108221	0.00075191	0.00085848
	2.0	0.00259763	0.01657506	0.00029103	0.00190229	0.00030350	0.00200435	0.00118276	0.00351316	0.00051825	0.00143293
	2.5	0.00039991	0.01194636	0.00081149	0.00144561	0.00086150	0.00142264	0.00129673	0.00399394	0.00037084	0.00222484
30	0.5	0.00046883	0.01847097	0.00085633	0.01936301	0.00172496	0.01918213	0.00032322	0.00140809	0.00032169	0.00137688
	1.0	0.00511380	0.01850442	0.00057205	0.00251467	0.00083788	0.00236932	0.00035554	0.00065701	0.00033709	0.00056284
	1.5	0.01110484	0.02048324	0.00018190	0.00131296	0.00019051	0.00128213	0.00044543	0.00020231	0.00042723	0.00015856
	2.0	0.00675664	0.01571503	0.00003949	0.00104566	0.00006733	0.00096870	0.00063681	0.00045794	0.00016368	0.00690045
	2.5	0.00150461	0.00863693	0.00045147	0.00075665	0.00038099	0.00072676	0.00062685	0.00015470	0.00011548	0.00960440
50	0.5	0.00017169	0.00866886	0.00015394	0.01016296	0.00054536	0.00977316	0.00019860	0.00093131	0.00040583	0.00050010
	1.0	0.00287558	0.00954984	0.00006641	0.00123665	0.00002331	0.00128988	0.00020136	0.00043550	0.00021472	0.00051113
	1.5	0.00583098	0.01021212	0.00000131	0.00066881	0.00000139	0.00061867	0.00025203	0.00010600	0.00022962	0.00006247
	2.0	0.00661202	0.00954337	0.00001165	0.00044830	0.00000984	0.00041441	0.00026438	0.00000487	0.00003089	0.01116049
	2.5	0.00059335	0.00299846	0.00039057	0.00030465	0.00040156	0.00032845	0.00022603	0.00026880	0.00003179	0.00959274

Table 5: MSE_s of MLE_s of θ and λ for fixed value of $\lambda = -0.5$.

n	θ	$\hat{\theta}_{ml}$	$\hat{\lambda}_{ml}$	$\hat{\mu}(0.5)$	$\hat{v}(0.5)$	$\hat{\mu}(1.0)$	$\hat{v}(1.0)$	$\hat{R}t_{ml}(0.5)$	$\hat{H}t_{ml}(0.5)$	$\hat{R}t_{ml}(1.0)$	$\hat{H}t_{ml}(1.0)$
10	0.5	0.00015920	0.00696553	0.13076640	0.00438970	0.14092320	0.00682297	0.00000703	0.00046831	0.00001291	0.00057327
	1.0	0.00229704	0.01036444	0.03029978	0.00315734	0.02791449	0.00209704	0.00039869	0.01021520	0.00031118	0.00875695
	1.5	0.00049121	0.00839531	0.01051839	0.00060575	0.00938667	0.00051351	0.00034847	0.01371874	0.00027858	0.01362306
	2.0	0.00432903	0.01992192	0.00820597	0.00000008	0.00808985	0.00001547	0.00022192	0.00790433	0.00003387	0.00010882
	2.5	0.01962041	0.03305859	0.00676651	0.00001499	0.00720680	0.00001094	0.00010565	0.00270725	0.00027221	0.00440608
15	0.5	0.00025608	0.00005568	0.04392407	0.00143653	0.05486502	0.00191518	0.00002924	0.00001680	0.00002018	0.00003355
	1.0	0.00182244	0.00227090	0.01674624	0.00195018	0.01774813	0.00194008	0.00006574	0.00484443	0.00008276	0.00505409
	1.5	0.00172535	0.00115507	0.00558294	0.00047894	0.00537249	0.00040485	0.00009314	0.01002115	0.00007994	0.00901803
	2.0	0.00000371	0.00435399	0.00352237	0.00003766	0.00422882	0.00003126	0.00014767	0.01094316	0.00001595	0.00417771
	2.5	0.00057183	0.00850315	0.00280693	0.00004041	0.00279106	0.00002950	0.00019980	0.01278288	0.00000593	0.00155778
20	0.5	0.00018980	0.00000832	0.03349772	0.00104132	0.03148375	0.00103393	0.00004374	0.00000001	0.00003356	0.00000076
	1.0	0.00164941	0.00038856	0.00951206	0.00131252	0.01086851	0.00119918	0.00000986	0.00272342	0.00000759	0.00276822
	1.5	0.00209499	0.00010919	0.00339258	0.00040016	0.00337497	0.00033631	0.00003303	0.00740499	0.00002462	0.00698175
	2.0	0.00000368	0.00209106	0.00270382	0.00002039	0.00262044	0.00003186	0.00006181	0.00758721	0.00002541	0.00446507
	2.5	0.00000389	0.00295920	0.00179154	0.00002552	0.00193241	0.00002726	0.00011165	0.01137240	0.00000011	0.00232036

30	0.5	0.00010174	0.00012969	0.01284169	0.00027226	0.01721395	0.00025279	0.00003181	0.00000625	0.00002711	0.00000347
	1.0	0.00078851	0.00011561	0.00607187	0.00074805	0.00517632	0.00092517	0.00000001	0.00119493	0.00013205	0.00364008
	1.5	0.00115521	0.00022143	0.00251279	0.00036002	0.00221007	0.00037046	0.00002871	0.00447934	0.00002304	0.00463863
	2.0	0.00030586	0.00041743	0.00130992	0.00003829	0.00111014	0.00001455	0.00003414	0.00551343	0.00002555	0.00429611
	2.5	0.00036518	0.00059613	0.00081997	0.00003221	0.00081823	0.00001746	0.00007246	0.00908388	0.00000395	0.00385952
50	0.5	0.00002529	0.00003850	0.00566839	0.00003747	0.00739973	0.00015790	0.00001368	0.00000559	0.00000868	0.00000090
	1.0	0.00074940	0.00004833	0.00219706	0.00102667	0.00086059	0.00000905	0.00000422	0.00073713	0.00000274	0.00004186
	1.5	0.00072863	0.00058762	0.00142703	0.00045568	0.00120045	0.00039972	0.00006114	0.00285978	0.00004945	0.00267791
	2.0	0.00004602	0.00013142	0.00055200	0.00000701	0.00045427	0.00001302	0.00000586	0.00178169	0.00001569	0.00177205
	2.5	0.00034990	0.00000157	0.00022482	0.00000297	0.00030510	0.00001031	0.00001130	0.00347030	0.00000429	0.00205595

Table 6: MSE_s of MPS_s of θ and λ for fixed value of $\lambda = -0.5$.

n	θ	$\hat{\theta}_{mp}$	$\hat{\lambda}_{mp}$	$\hat{\mu}(0.5)$	$\hat{v}(0.5)$	$\hat{\mu}(1.0)$	$\hat{v}(1.0)$	$\hat{R}t_{mp}(0.5)$	$\hat{H}t_{mp}(0.5)$	$\hat{R}t_{mp}(1.0)$	$\hat{H}t_{mp}(1.0)$
10	0.5	0.01348169	0.33025170	2.84564600	0.00025662	2.88879000	0.00000059	0.00071394	0.00255441	0.00076153	0.00276055
	1.0	0.04582863	0.34362310	0.59848730	0.00022376	0.57886200	0.00002467	0.00198911	0.00642987	0.00180874	0.00548711
	1.5	0.12441630	0.32601920	0.22689160	0.00000818	0.22622710	0.00001829	0.00123520	0.00037053	0.00119271	0.00032896
	2.0	0.28729430	0.35836700	0.13626710	0.00050645	0.13401370	0.00070230	0.00027906	0.00913453	0.00157940	0.10502300
	2.5	0.51065350	0.39146690	0.08992571	0.00053008	0.09171286	0.00051683	0.00000251	0.05403420	0.00296747	0.25762670
15	0.5	0.00831857	0.19392400	1.63104900	0.00001158	1.69951200	0.00000042	0.00038272	0.00136624	0.00040088	0.00143433
	1.0	0.03053118	0.22381810	0.37937380	0.00048060	0.38105350	0.00047463	0.00123228	0.00400713	0.00126594	0.00415574
	1.5	0.07652181	0.21094650	0.14365410	0.00005367	0.14639380	0.00003739	0.00087977	0.00062139	0.00085409	0.00044673
	2.0	0.16117420	0.22107990	0.07608407	0.00002230	0.08171975	0.00003120	0.00034785	0.00253914	0.00054055	0.05241439
	2.5	0.27331720	0.23828870	0.04744720	0.00000759	0.04915973	0.00001248	0.00006421	0.01922894	0.00112445	0.12974360
20	0.5	0.00574167	0.13224940	1.11737200	0.00002113	1.11422900	0.00003138	0.00023905	0.00087760	0.00025294	0.00091724
	1.0	0.02041360	0.14919280	0.24888810	0.00051379	0.26038200	0.00041758	0.00080515	0.00272592	0.00080533	0.00271679
	1.5	0.05149830	0.14521440	0.09539313	0.00012748	0.09509341	0.00008451	0.00065142	0.00062191	0.00060252	0.00052410
	2.0	0.11504040	0.15453840	0.05229723	0.00000722	0.05012395	0.00000319	0.00023156	0.00183693	0.00031909	0.03351163
	2.5	0.18583370	0.16112860	0.03104992	0.00000036	0.03136896	0.00000031	0.00005039	0.01248162	0.00070330	0.08613870
30	0.5	0.00327063	0.07029668	0.56871670	0.00000508	0.60889400	0.00000315	0.00012112	0.00044604	0.00012834	0.00046661
	1.0	0.01220098	0.08533019	0.14259940	0.00039385	0.13717000	0.00052455	0.00041794	0.00147061	0.00033599	0.00001755
	1.5	0.03013804	0.08933523	0.05774618	0.00021172	0.05415080	0.00020952	0.00044174	0.00060492	0.00042327	0.00062699
	2.0	0.05882264	0.08119187	0.02629523	0.00000491	0.02567915	0.00000004	0.00015231	0.00059272	0.00015596	0.01717963
	2.5	0.09574786	0.08500667	0.01538689	0.00000757	0.01553761	0.00000136	0.00004603	0.00548359	0.00033139	0.04254537
50	0.5	0.00156213	0.03111785	0.24210070	0.00000037	0.24455010	0.00002632	0.00005097	0.00018673	0.00005838	0.00021613
	1.0	0.00425422	0.03733464	0.05810634	0.00082386	0.0286866	0.00007953	0.00023847	0.00099346	0.03189693	0.24571120
	1.5	0.01263423	0.04526659	0.02565940	0.00038205	0.02455903	0.00032532	0.00033951	0.00074756	0.00030989	0.00065741
	2.0	0.02853440	0.03613592	0.01101037	0.00000053	0.01026380	0.00000305	0.00005461	0.00053151	0.00006510	0.00833448
	2.5	0.04052860	0.03267774	0.00580179	0.00000011	0.00592527	0.00000273	0.00000746	0.00298680	0.00014217	0.01924047

Conclusion

In the present piece of work, we have proposed a new lifetime distribution and called it as $TD_E(\theta)$-distribution. The proposed distribution is more useful to analyze the real life data with non-monotone failure rate. We derived its different statistical properties such as moments, survival function, hazard function, quantile function, and order statistics etc. It has been shown that the $TD_E(\theta)$-distribution provide better fit as compared to the baseline distribution and to some other distributions which are $TIWD, IWD, TIED, TIRD$ and $DUS_E(\theta)$ $-$distribution. Thus, we may conclude that our proposed $TD_E(\theta)$-distribution is more flexible and can be considered as a suitable model for a large variety of lifetime data, and particularly it is more useful to analyze the real data having non-monotone failure rate.

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