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On Generalized Minty and Stampacchia Vector Variational-Like Inequalities and Nonsmooth Vector Optimization Problem Involving Higher Order Strong Invexity

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***Abstract*—This paper deals with the relations between generalized Minty vector variational-like inequality problems, generalized Stampacchia vector variational-like inequality problems and a class of nonsmooth vector optimization problems by using the concept of an efficient minimizer of order m in terms of Clarke subdifferential. Furthermore, we consider weak formulations of considered generalized Minty and Stampacchia vector variational-like inequality problems and establish the relationship between the solution of these vector variational-like inequality problems and strict minimizer of order m of considered nonsmooth vector optimization problem. Moreover, we employ KKM-Fan theorem to establish some existence results for the solutions of the generalized Minty and Stampacchia vector variational-like inequality problems. The results of the paper extend and unify some earlier results of Bhatia (2008), Li & Yu (2017) and Upadhyay et al. (2017) to the nondifferentiable case as well as for a more general class of nonconvex functions.**

***Index Terms*—Efficient minimizers of order m, KKM-Fan theorem, locally Lipschitz functions, strict minimizers of order m, vector variational-like inequality problems.**

I. Introduction

It is well known that convexity and generalized convexity have wider applications in optimization theory, engineering, economics, probability theory and calculus of variations, see Green & Heller (1981); Rahtu et al. (2006); Smith (1985) and the references cited therein. Mangasarian (1969) introduced the concept of pseudoconvex functions as a generalization of convex functions. Karamardian & Schiable (1990) introduced the class of strongly convex functions of order 2, which was later generalized by Lin & Fukushima (2003) as a strongly convex function of order m. Hanson (1981) introduced the concept of invex functions as a generalization of convex functions. Invex functions possess several properties, for example, a critical point is global minima and first order necessary optimality conditions become sufficient, which led to the various applications of invex functions in nonlinear optimization and variational inequality problems, see Ben-Israel & Mond (1986); Mishra & Upadhyay (2015); Weir & Mond (1988) and the references cited therein. Kaul & Kaur (1985) defined the concept of pseudoinvex and quasiinvex functions to obtain sufficient optimality criteria for nonlinear programming problems involving these functions. Jeyakumar & Mond (1992) introduced the concept of strongly α- invex functions. Reiland (1990) extended the notion of invexity for nonsmooth functions with the help of generalized gradient introduced by Clarke (1983).

In vector optimization problems, efficiency is a widely used solution concept. Since in numerical techniques, algorithms terminate after certain steps, so we get only approximate solutions. The concept of approximate solutions can be considered as a satisfactory compromise to the efficient values of the objective of a vector optimization problem with some relative error, for details see Deng (1997); Gupta & Mehra (2008) and references cited therein. Cromme (1978) studied the concept of strict local minimizers while studying the convergence of iterative numerical techniques. Auslender (1984) and Ward (1994) extended the concept of strict local minimizer to a strict local minimizer of order m. Jiménez (2002) introduced the notion of a strict minimizer of higher order for vector optimization problems.

Variational inequality was first introduced by Hartman & Stampacchia (1980) as a tool for the study of some specific classes of partial differential equations. Variational inequalities

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are either known as Stampacchia variational inequalities introduced by Stampacchia (1960) or in the form of Minty variational inequalities introduced by Minty (1967). The notion of vector variational inequality was introduced by Giannessi (1980) in finite dimensions. In literature, most of the scholars discussed applications of Stampacchia vector variational inequality and the Minty vector variational inequality for the vector optimization problem, for more expositions, see Ansari & Siddiqi (1998); Ansari & Yao (2000); Bhatia (2008); Giannessi (1998, 2000); Lee (2000); Mishra & Upadhyay (2013); Upadhyay et al. (2019) and the references cited therein. Li & Yu (2017) established the relationship between solutions of vector variational inequalities and vector optimization problem for directionally differentiable invex functions. Mishra & Wang (2006) established the relationship between nonsmooth vector optimization problem and vector variational-like inequality problems under nonsmooth invexity. Al-Homidan & Ansari (2010) gave such results for weak efficient solution of the nonsmooth vector optimization problem. Oveisiha & Zafarani (2013) established the relationship between vector variational-like inequality problems and nonsmooth vector optimization problems using α-invex function in Asplund spaces with limiting subdifferential.

Motivated by the works of Al-Homidan & Ansari (2010), Li & Yu (2017), Upadhyay et al. (2017) and Mishra & Wang (2006), we consider generalized Minty and Stampacchia vector variational-like inequality problems and a class of nonsmooth vector optimization problems. We establish the relationship between the solutions of generalized Minty and Stampacchia vector variational-like inequality problems and efficient minimizer of order m of nonsmooth vector optimization problems by using the concepts of strong invexity of order m. We also discuss the weak formulation of generalized Minty and Stampacchia vector variational-like inequality problems and establish the relationship between strict minimizer of order m of vector optimization problems and solutions of weak generalized vector variational-like inequality problems. Furthermore, we employ KKM-Fan theorem to establish the existence of a solution of generalized Minty and Stampacchia vector variational-like inequality problems.

This paper is organized as follows. In Section II, some definitions are given which will be used throughout the paper. In Section III, we establish the relationship between the solution of generalized Minty and Stampacchia vector variational-like inequality problems and efficient minimizer of order m of nonsmooth vector optimization problems for strongly invex functions of order m. In Section IV, weak generalized Minty and Stampacchia vector variational-like inequality problems are considered and we establish the relationship between the solution of weak generalized Minty and Stampacchia vector variational-like inequality problems with strict minimizer of order m of the vector optimization problem. In Section V, we discuss the existence of solution of generalized vector variational-like inequality problems with the help of KKM-Fan theorem.

II. Definitions and preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $-\mathbb{R}^n_+$ denotes its nonpositive orthant. $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^n$. Interior of $\mathbb{R}^n$ is denoted by $\text{int}\mathbb{R}^n$ and let $\langle .,. \rangle$ denotes the Euclidean inner product. Let $\Gamma \subseteq \mathbb{R}^n$ be a nonempty set equipped with Euclidean norm $||.||$. Let $\eta : \Gamma \times \Gamma \to \mathbb{R}^n$ be a vector valued function.

For $x, y \in \mathbb{R}^n$, the following convention for equalities and inequalities will be used throughout the paper.

1) $x - y \in -\mathbb{R}^n_+ \iff x_i \leq y_i,\ i = 1, 2, \ldots, n;$
2) $x - y \in -\mathbb{R}^n_+ \setminus \{\mathbf{0}\} \iff x_i \leq y_i,\ i = 1, 2, \ldots, n$ with strict inequality for at least one i;
3) $x - y \in -\text{int}\mathbb{R}^n_+ \iff x_i < y_i,\ i = 1, 2, \ldots, n.$

The following notions of nonsmooth analysis are from Clarke (1983).

Definition 1: A function $f : \Gamma \to \mathbb{R}$ is said to be Lipschitz near $x \in \Gamma$, if there exists a positive constant K and a neighbourhood N of x, such that, for any $y, z \in N$, one has

$$|f(y) - f(z)| \leq K||y - z||.$$

The function f is locally Lipschitz on Γ, if it is Lipschitz near x, for every $x \in \Gamma$.

Definition 2: Let $f : \Gamma \to \mathbb{R}$ be locally Lipschitz at a given point $x \in \Gamma$. The Clarke generalized directional derivative of f at $x \in \Gamma$, in the direction $v \in \mathbb{R}^n$, denoted by $f^\circ(x; v)$, is defined as

$$f^\circ(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

Definition 3: Let $f : \Gamma \to \mathbb{R}$ be locally Lipschitz at a given point $x \in \Gamma$. The Clarke generalized subdifferential of f at $x \in \Gamma$, denoted by $\partial^c f(x)$, is defined as

$$\partial^c f(x) := \{\xi \in \mathbb{R}^n : f^\circ(x; v) \geq \langle \xi, v \rangle,\ \ \forall v \in \mathbb{R}^n\}.$$

The following definitions are from Al-Homidan & Ansari (2010).

Definition 4: Let x be any arbitrary point of Γ. The set Γ is said to be invex at x with respect to η if, for all $y \in \Gamma$

$$x + \lambda\eta(y, x) \in \Gamma,\ \ \forall \lambda \in [0, 1].$$

The set Γ is said to be an invex set with respect to η, if Γ is invex at every point $x \in \Gamma$ with respect to η.

The following notions of strong invexity and monotonicity are from Jabarootian & Zafarani (2006).

Definition 5: Let Γ be an invex set with respect to η. A function $f : \Gamma \to \mathbb{R}$ is said to be strongly preinvex of order $m \geq 1$ with respect to η on Γ, if there exists a constant $c > 0$, such that, for all $x, y \in \Gamma$ and $t \in [0, 1]$, one has

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y) - t(1 - t)c||\eta(y, x)||^m.$$

Definition 6: A function $f : \Gamma \to \mathbb{R}$ is said to be strongly invex of order $m \geq 1$ with respect to η on Γ, if there exists a constant $c > 0$, such that, for all $x, y \in \Gamma$, one has

$$f(x) - f(y) \geq \langle \xi, \eta(x, y) \rangle + c||\eta(x, y)||^m,\ \ \forall \xi \in \partial^c f(y).$$

Example 1: Let $\Gamma = [-1, 1]$ and $f : \Gamma \to \mathbb{R},\ \ \eta : \Gamma\times\Gamma \to \mathbb{R}$ given by-

$$f = \begin{cases} x^2 - 1, & \text{if} \ \ x \geq 0, \\ x^2 - x - 1, & \text{if} \ \ x < 0, \end{cases}$$

$$\text{and } \eta(x, y) = \begin{cases} (x - y)^{1/2}, & \text{if } x \geq 0 \ and \ y < 0, \\ x - y, & \text{elsewhere.} \end{cases}$$

It can be verified that f is a strongly invex function of order $m = 2$ with $c = 1$.

Definition 7: Let $T : \Gamma \to 2^{\Gamma}$ be a set valued map. T is said to be strongly invariant monotone of order m, with respect to η on Γ, if there exists a constant $c > 0$, such that, for any $x, y \in \Gamma$, and any $\xi \in T(x)$, $\zeta \in T(y)$, one has

$$\langle \xi, \eta(y, x)\rangle + \langle \zeta, \eta(x, y)\rangle \leq -c\{||\eta(y, x)||^m + ||\eta(x, y)||^m\}.$$

Condition A. Yang et al. (2003) Let Γ be an invex set with respect to η. Then the function $f : \Gamma \to \mathbb{R}$ is said to be satisfy the Condition A, if

$$f(y + \eta(x, y)) \leq f(x), \ \ \forall x, y \in \Gamma.$$

Condition C. Mohan & Neogy (1995) Let Γ be an invex set with respect to η. Then, η is said to be satisfy the Condition C if, for all $x, y \in \Gamma,\ \ \lambda_1, \lambda_2, \lambda \in [0, 1]$, one has

(i) $\eta(x, x + \lambda\eta(y, x)) = -\lambda\eta(y, x)$,
(ii) $\eta(y, x + \lambda\eta(y, x)) = (1 - \lambda)\eta(y, x)$.

Remark 1: Yang et al. (2003) have shown that

$$\eta(x + \lambda_1\eta(y, x), x + \lambda_2\eta(y, x)) = (\lambda_1 - \lambda_2)\eta(y, x).$$

The map $\eta(x, \tilde{x}) = x - \tilde{x}$ satisfies all the conditions trivially. For a nontrivial example of η, satisfying all the above conditions, we refer to Al-Homidan & Ansari (2010).

Now, we state the following lemma from Jabarootian & Zafarani (2006), which establishes the relationship between strongly invex function of order m and strongly invariant monotonicity of order m of its generalized gradient.

Lemma 1: Let Γ be an invex set with respect to η and $f : \Gamma \to \mathbb{R}$ be locally Lipschitz on Γ. If f is strongly invex of order m with respect to η, then $\partial^c f$ is strongly invariant monotone of order m with respect to η, that is, for all $x, y \in \Gamma$, $\xi \in \partial^c f(x)$ and $\zeta \in \partial^c f(y)$,

$$\langle \xi, \eta(y, x)\rangle + \langle \zeta, \eta(x, y)\rangle \leq -c\{||\eta(y, x)||^m + ||\eta(x, y)||^m\}.$$

The following Lebourg mean value theorem from Clarke (1983) will be used in the sequel.

Theorem 1: Let x and y be points in Γ, and suppose that f is Lipschitz on an open set containing the line segment $[x, y]$. Then, there exists a point u in (x, y), such that

$$f(y) - f(x) \in \langle \partial^c f(u), y - x\rangle.$$

Now, we prove the following lemma, which generalizes the corresponding result from Jabarootian & Zafarani (2006) for any $m > 0$.

Lemma 2: Let Γ be an invex set with respect to η such that η satisfy the Condition C and let $f : \Gamma \to \mathbb{R}$ be locally Lipschitz on Γ. If the function f is strongly invex of order m with respect to η on Γ, then f is strongly preinvex of order m with respect to the same η on Γ.

Proof: Let $\tilde{x} = y + \lambda\eta(x, y),\ \lambda \in [0, 1]$. Since Γ is invex, $\tilde{x} \in \Gamma$. Since, f be a strongly invex function of order m with respect to η on Γ. Then, we have

$$f(y) - f(\tilde{x}) \geq \langle \xi, \eta(y, \tilde{x})\rangle + c||\eta(y, \tilde{x})||^m$$

From the Condition C, we get

$$f(y) - f(\tilde{x}) \geq -\lambda\langle \xi, \eta(x, y)\rangle + c\lambda^m||\eta(x, y)||^m. \quad (1)$$

Similarly, we have

$$\begin{aligned} f(x) - f(\tilde{x}) &\geq \langle \xi, \eta(x, \tilde{x})\rangle + c||\eta(x, \tilde{x})||^m \\ &= (1 - \lambda)\langle \xi, \eta(x, y)\rangle + c(1 - \lambda)^m||\eta(x, y)||^m. \end{aligned} \quad (2)$$

Multiplying Eq. (1) by $1 - \lambda$ and Eq. (2) by λ and adding the resulting inequalities, we get

$$\begin{aligned} &(1 - \lambda)f(y) + \lambda f(x) - f(\tilde{x}) \\ &\geq c[\lambda^m(1 - \lambda) + \lambda(1 - \lambda)^m]||\eta(x, y)||^m. \end{aligned} \quad (3)$$

If $0 < m \leq 2$, then

$$(1 - \lambda)^{m-1} + \lambda^{m-1} \geq (1 - \lambda) + \lambda = 1. \quad (4)$$

If $m > 2$, then $\phi(\lambda) = \lambda^{m-1}$ is convex on $(0, 1)$, therefore,

$$(1 - \lambda)^{m-1} + \lambda^{m-1} \geq (\frac{1}{2})^{m-2}. \quad (5)$$

Therefore, from Eq. (3), Eq. (4), and Eq. (5), there exist $\bar{c} > 0$ independent of $x, y,$ and λ, such that

$$f(y+\lambda\eta(x, y)) \leq (1-\lambda)f(y)+\lambda f(x)-\bar{c}\lambda(1-\lambda)||\eta(x, y)||^m.$$

Hence, f is strongly preinvex function of order m with respect to η. ■

We consider the following nonsmooth vector optimization problem:

$$\text{(NVOP)} \qquad \begin{array}{ll} \text{Minimize} & f(x) = (f_1(x), \ldots, f_p(x)) \\ \text{subject to} & x \in \Gamma, \end{array}$$

where $f_i : \Gamma \to \mathbb{R}, i \in \mathcal{I} := \{1, 2, ..., p\}$ are non-differentiable locally Lipschitz functions on Γ.

The following notions of efficient minimizer and strict minimizer of order m with respect to η of (NVOP) are from Upadhyay et al. (2017).

Definition 8: Let $m \geq 1$ be an integer. A point $y \in \Gamma$ is said to be an efficient minimizer of order m with respect to η of (NVOP), if there exists a constant $c \in \text{int}\mathbb{R}^p_+$, such that

$$\begin{aligned} &(f_1(x) - f_1(y) - c_1||\eta(x, y)||^m, \ldots, \\ &\quad f_p(x) - f_p(y) - c_p||\eta(x, y)||^m) \notin -\mathbb{R}^p_+ \setminus \{0\}, \ \ \forall x \in \Gamma. \end{aligned}$$

Definition 9: Let $m \geq 1$ be an integer. A point $y \in \Gamma$ is said to be a strict minimizer of order m with respect to η of (NVOP), if there exists a constant $c \in \text{int}\mathbb{R}^p_+$, such that

$$\begin{aligned} &(f_1(x) - f_1(y) - c_1||\eta(x, y)||^m, ..., \\ &\quad f_p(x) - f_p(y) - c_p||\eta(x, y)||^m) \notin -\text{int}\mathbb{R}^p_+, \ \ \forall x \in \Gamma. \end{aligned}$$

Remark 2: It is obvious from the above definitions that every efficient minimizer of order m with respect to η is also a strict minimizer of order m with respect to η of (NVOP), but the converse may not be true. To illustrate this fact, we consider the following nonsmooth vector optimization problem

$$\text{(P1)} \qquad \text{Minimize} \quad f(x) = (f_1(x), f_2(x)) \qquad \text{subject to} \quad x \in \Gamma,$$

where $\Gamma = [-1, 1]$, $f = (f_1, f_2) : [-1, 1] \to \mathbb{R}^2$ and $\eta : [-1, 1] \times [-1, 1] \to \mathbb{R}^2$ be defined as:

$$f_1 = \begin{cases} x^2 - 2x, & x \geq 0, \\ e^{-x} - 1, & x < 0, \end{cases} \qquad f_2 = \begin{cases} x^3 + 1, & x \geq 0, \\ e^{-x}, & x < 0, \end{cases}$$

$$\text{and} \quad \eta(x, y) = \begin{cases} x - y, & x \geq 0,\, y \geq 0, \text{ or } x < 0,\, y < 0, \\ y - x, & x \geq 0,\, y < 0, \text{ or } x < 0,\, y \geq 0. \end{cases}$$

It is not hard to verify that $y = 0$ is a strict minimizer of order 3 with $c = (1, 1)$, but not an efficient minimizer of order 3 with respect to η.

Now, we consider the following generalized Minty and Stampacchia vector variational-like inequality problems from Al-Homidan & Ansari (2010), in terms of Clarke subdifferential:

(GMVVLIP) A generalized Minty vector variational-like inequality problem is to find $\tilde{x} \in \Gamma$, such that, for each $y \in \Gamma$ and for all $\xi_i \in \partial^c f_i(y)$, $i \in \mathcal{I}$, we have

$$\langle \xi, \eta(y, \tilde{x}) \rangle_p = (\langle \xi_1, \eta(y, \tilde{x}) \rangle, \ldots, \langle \xi_p, \eta(y, \tilde{x}) \rangle) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}.$$

(GSVVLIP) A generalized Stampacchia vector variational-like inequality problem is to find $\tilde{x} \in \Gamma$, such that, for each $y \in \Gamma$, there exists $\zeta_i \in \partial^c f_i(\tilde{x}), i \in \mathcal{I}$, we have

$$\langle \zeta, \eta(y, \tilde{x}) \rangle_p = (\langle \zeta_1, \eta(y, \tilde{x}) \rangle, \ldots, \langle \zeta_p, \eta(y, \tilde{x}) \rangle) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}.$$

III. Relationship between (GMVVLIP), (GSVVLIP) and (NVOP)

In this section, using the tools of nonsmooth analysis and notion of strong invexity of order m, we establish certain relations between the solutions of generalized Minty and Stampacchia vector variational-like inequality problems (GMVVLIP), (GSVVLIP) and the efficient minimizer of order m with respect to η of nonsmooth vector optimization problem (NVOP).

Theorem 2: Let Γ be an invex set with respect to η, such that η is skew and satisfies the Condition C. Let each f_i, $i \in \mathcal{I}$ be strongly invex function of order m with respect to η on Γ and satisfy the Condition A. Then $\tilde{x} \in \Gamma$ is an efficient minimizer of order m with respect to η of (NVOP) if and only if $\tilde{x}$ is a solution of (GMVVLIP).

Proof: Let $\tilde{x} \in \Gamma$ be an efficient minimizer of order m with respect to η of (NVOP). Then, for all $x \in \Gamma$, there exists a constant $c \in \text{int}\mathbb{R}^p_+$, such that

$$(f_1(x) - f_1(\tilde{x}) - c_1||\eta(x, \tilde{x})||^m, \ldots, f_p(x) - f_p(\tilde{x}) - c_p||\eta(x, \tilde{x})||^m) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}. \quad (6)$$

Since, each f_i, $i \in \mathcal{I}$ is strongly invex of order m with respect to η, therefore for all $x \in \Gamma$ and $\xi_i \in \partial^c f_i(x)$, we have

$$f_i(\tilde{x}) - f_i(x) \geq \langle \xi_i, \eta(\tilde{x}, x) \rangle + c_i||\eta(\tilde{x}, x)||^m, \quad \forall i \in \mathcal{I}. \quad (7)$$

Since η is skew, from Eq. (7), we get

$$f_i(x) - f_i(\tilde{x}) \leq \langle \xi_i, \eta(x, \tilde{x}) \rangle - c_i||\eta(x, \tilde{x})||^m, \quad \forall i \in \mathcal{I}. \quad (8)$$

Since $c_i > 0$, from Eq. (8), we have

$$f_i(x) - f_i(\tilde{x}) - c_i||\eta(x, \tilde{x})||^m \leq \langle \xi_i, \eta(x, \tilde{x}) \rangle - 2c_i||\eta(x, \tilde{x})||^m, \quad \forall i \in \mathcal{I}. \quad (9)$$

From Eq. (6) and Eq. (9), for all $x \in \Gamma$, we get

$$(\langle \xi_1, \eta(x, \tilde{x}) \rangle, ..., \langle \xi_p, \eta(x, \tilde{x}) \rangle) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}.$$

Hence $\tilde{x}$ is a solution of (GMVVLIP).

Conversely, Let $\tilde{x} \in \Gamma$ be a solution of (GMVVLIP), but not an efficient minimizer of order m with respect to η of (NVOP). Then, there exist $x \in \Gamma$, such that for any $c \in \text{int}\mathbb{R}^p_+$, we have

$$(f_1(x) - f_1(\tilde{x}) - c_1||\eta(x, \tilde{x})||^m, \ldots, f_p(x) - f_p(\tilde{x}) - c_p||\eta(x, \tilde{x})||^m) \in -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}. \quad (10)$$

Let $x(t) := \tilde{x} + t\eta(x, \tilde{x})$ for all $t \in [0, 1]$. Since Γ is invex set, $x(t) \in \Gamma$, for all $t \in [0, 1]$.

Choose $t' \in (0, 1)$ arbitrary. By Lemma 2, each f_i, $i \in \mathcal{I}$ is strongly preinvex of order m with respect to η. Therefore, there exists $c \in \text{int}\mathbb{R}^p_+$, such that for all $i \in \mathcal{I}$, we have

$$f_i(x(t')) = f_i(\tilde{x} + t'\eta(x, \tilde{x})) \leq (1 - t')f_i(\tilde{x}) + t'f_i(x) - t'(1 - t')c_i||\eta(x, \tilde{x})||^m. \quad (11)$$

From Eq. (11), it follows that

$$f_i(\tilde{x} + t'\eta(x, \tilde{x})) - f_i(\tilde{x}) \leq t'[f_i(x) - f_i(\tilde{x}) - (1 - t')c_i||\eta(x, \tilde{x})||^m], \quad \forall i \in \mathcal{I}. \quad (12)$$

From the mean value Theorem 1, there exists $t_i \in (0, t')$ and $\xi_i \in \partial^c f_i(x(t_i))$, where $x(t_i) = \tilde{x} + t_i\eta(x, \tilde{x})$, we have

$$t'\langle \xi_i, \eta(x, \tilde{x}) \rangle = f_i(\tilde{x} + t'\eta(x, \tilde{x})) - f_i(\tilde{x}), \quad \forall\, i \in \mathcal{I}. \quad (13)$$

From Eq. (12) and Eq. (13), we have

$$\langle \xi_i, \eta(x, \tilde{x}) \rangle \leq f_i(x) - f_i(\tilde{x}) - (1 - t')c_i||\eta(x, \tilde{x})||^m, \ \forall i \in \mathcal{I}. \quad (14)$$

Suppose that $t_1, t_2, ..., t_p$ are all equal to t. From Eq. (10) and Eq. (14), it follows that

$$(\langle \xi_1, \eta(x, \tilde{x}) \rangle, ..., \langle \xi_p, \eta(x, \tilde{x}) \rangle) \in -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}. \quad (15)$$

Then from the Condition C, we have

$$\langle \xi_i, \eta(x(t), \tilde{x}) \rangle = t\langle \xi_i, \eta(x, \tilde{x}) \rangle, \quad \forall i \in \mathcal{I} \quad (16)$$

From Eq. (15) and Eq. (16), it follows that

$$(\langle \xi_1, \eta(x(t), \tilde{x}) \rangle, ..., \langle \xi_p, \eta(x(t), \tilde{x}) \rangle) \in -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}.$$

This implies that $\tilde{x}$ does not solve (GMVVLIP). This contradicts our assumption.

Now, consider the case $t_1, t_2, \ldots, t_p$ are not all equal. Let

$t_i \neq t_j$, for some $i, j \in \mathcal{I}$, $\xi_i \in \partial^c f_i(x(t_i)), \xi_j \in \partial^c f_j(x(t_j))$, and $i \neq j$, then from Eq. (14), we have

$$\langle \xi_i, \eta(x, \tilde{x}) \rangle \leq f_i(x) - f_i(\tilde{x}). \quad (17)$$

$$\langle \xi_j, \eta(x, \tilde{x}) \rangle \leq f_j(x) - f_j(\tilde{x}). \quad (18)$$

Since f_i and f_j are strongly invex of order m with respect to η, therefore, from Lemma 1, $\partial^c f_i$ and $\partial^c f_j$ are strongly invariant monotone of order m with respect to η. Hence, for all $\xi_i \in \partial^c f_i(x(t_i))$ and $\xi_j^* \in \partial^c f_i(x(t_j))$, we get

$$\langle \xi_i, \eta(x(t_j), x(t_i)) \rangle + \langle \xi_j^*, \eta(x(t_i), x(t_j)) \rangle \leq -c_i[||\eta(x(t_i), x(t_j))||^m + \eta(x(t_j), x(t_i))||^m]. \quad (19)$$

Since η is skew, from Eq. (19), it follows that

$$\langle \xi_i - \xi_j^*, \eta(x(t_i), x(t_j)) \rangle \geq c_i||\eta(x(t_i), x(t_j))||^m, \quad \forall\, \xi_j^* \in \partial^c f_i(x(t_j)). \quad (20)$$

Similarly,

$$\langle \xi_i^* - \xi_j, \eta(x(t_i), x(t_j)) \rangle \geq c_j||\eta(x(t_i), x(t_j))||^m, \quad \forall\, \xi_i^* \in \partial^c f_j(x(t_i)). \quad (21)$$

If $t_i > t_j$, then by Remark 1 and Eq. (20), there exist $\bar{c_i} > 0$, such that

$$\bar{c_i}||\eta(x, \tilde{x})||^m \leq \langle \xi_i - \xi_j^*, \eta(x, \tilde{x}) \rangle. \quad (22)$$

From Eq. (22), it follows that

$$\langle \xi_i, \eta(x, \tilde{x}) \rangle \geq \langle \xi_j^*, \eta(x, \tilde{x}) \rangle + \bar{c_i}||\eta(x, \tilde{x})||^m, \quad \text{where } \bar{c_i} = (t_i - t_j)^{m-1}.$$

From Eq. (17), we have

$$f_i(x) - f_i(\tilde{x}) \geq \langle \xi_j^*, \eta(x, \tilde{x}) \rangle + \bar{c_i}||\eta(x, \tilde{x})||^m, \quad \forall \xi_j^* \in \partial^c f_i(x(t_j)).$$

If $t_i < t_j$, then by Remark 1 and Eq. (21), we get

$$\bar{c_j}||\eta(x, \tilde{x})||^m \leq \langle \xi_j - \xi_i^*, \eta(x, \tilde{x}) \rangle. \quad (23)$$

From Eq. (23), we have

$$\langle \xi_j, \eta(x, \tilde{x}) \rangle \geq \langle \xi_i^*, \eta(x, \tilde{x}) \rangle + \bar{c_j}||\eta(x, \tilde{x})||^m, \quad \text{where } \bar{c_j} = (t_j - t_i)^{m-1}.$$

From Eq. (18), it follows that

$$f_j(x) - f_j(\tilde{x}) \geq \langle \xi_i^*, \eta(x, \tilde{x}) \rangle + \bar{c_i}||\eta(x, \tilde{x})||^m, \quad \text{for any } \xi_i^* \in \partial^c f_j(x(t_i)).$$

Let, $\bar{t} = \min\{t_i, t_j\}$, we can get $\bar{\xi_k} \in \partial^c f_k(x(\bar{t}))$, such that

$$\langle \bar{\xi_k}, \eta(x, \tilde{x}) \rangle \leq f_k(x) - f_k(\tilde{x}) - \bar{c_k}||\eta(x, \tilde{x})||^m, \quad \text{for any } k = i, j.$$

Continuing in this way, there exists $\xi_i^* \in \partial^c f_i(x(t^*))$, such that $t^* = \min\{t_1, t_2, ..., t_p\}$ and

$$\langle \xi_i^*, \eta(x, \tilde{x}) \rangle \leq f_i(x) - f_i(\tilde{x}) - \bar{c_i}||\eta(x, \tilde{x})||^m, \ \ \forall\, i \in \mathcal{I}. \quad (24)$$

From Eq. (10) and Eq. (24), we get $\xi_i^* \in \partial^c f_i(x(t^*)),\ i \in \mathcal{I}$,

$$(\langle \xi_1^*, \eta(x, \tilde{x}) \rangle, ..., \langle \xi_p^*, \eta(x, \tilde{x}) \rangle) \in -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}. \quad (25)$$

Since η is skew, multiplying Eq. (25) by $-t^*$ and using the Condition C, we get

$$(\langle \xi_1^*, \eta(x(t^*), \tilde{x}) \rangle, ..., \langle \xi_p^*, \eta(x(t^*), \tilde{x}) \rangle) \in -\mathbb{R}^p_+ \setminus \{\mathbf{0}\},$$

which is a contradiction. ■

Theorem 3: Let Γ be an invex set with respect to η. Let each f_i, $i \in \mathcal{I}$ be strongly invex function of order m with respect to η on Γ. If $\tilde{x} \in \Gamma$ be a solution of (GSVVLIP), then $\tilde{x}$ is an efficient minimizer of order m with respect to η of (NVOP).

Proof: Let $\tilde{x} \in \Gamma$ be a solution of (GSVVLIP), for any $y \in \Gamma$, there exists $\zeta_i \in \partial^c f_i(\tilde{x}), i \in \mathcal{I}$, such that

$$(\langle \zeta_1, \eta(y, \tilde{x}) \rangle, \ldots, \langle \zeta_p, \eta(y, \tilde{x}) \rangle) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}. \quad (26)$$

Since each f_i, $i \in \mathcal{I}$ is strongly invex function of order m with respect to η, there exists a constant $c \in \text{int}\mathbb{R}^p_+$, such that

$$\langle \zeta_i, \eta(y, \tilde{x}) \rangle + c_i||\eta(y, \tilde{x})||^m \leq f_i(y) - f_i(\tilde{x}), \ \ \forall y \in \Gamma. \quad (27)$$

From Eq. (26) and Eq. (27), we have

$$(f_1(y) - f_1(\tilde{x}) - c_1||\eta(y, \tilde{x})||^m, ..., f_p(y) - f_p(\tilde{x}) - c_p||\eta(y, \tilde{x})||^m) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}.$$

Hence, $\tilde{x} \in \Gamma$ is an efficient minimizer of order m with respect to η of (NVOP). ■

Remark 3: The converse of the Theorem 3 may not hold. For example, consider the following nonsmooth vector optimization problem

$$\text{(P2)} \qquad \begin{array}{ll} \text{Minimize} & f(x) = (f_1(x), f_2(x)) \\ \text{subject to} & x \in \Gamma, \end{array}$$

where, $f = (f_1, f_2) : [-1, 1] \to \mathbb{R}^2$ and $\eta : [-1, 1] \times [-1, 1] \to \mathbb{R}^2$ be defined as:

$$f_1 = \begin{cases} x^2 - x, & x \geq 0, \\ x^2 - 2x, & x < 0, \end{cases} \quad f_2 = \begin{cases} x^2 + e^x, & x \geq 0, \\ x^2 + e^{-x}, & x < 0, \end{cases}$$

$$\text{and } \ \eta(x, y) = \begin{cases} 1 - y, & x \geq 0 \text{ and } y < 0, \\ x - y, & \text{elsewhere.} \end{cases}$$

It can be verified that the functions f_i, $i = 1, 2$ are strongly invex functions of order 2 with $c_i = 1$, $i = 1, 2$ with respect to η on $[-1, 1]$. Now, we can evaluate that

$$\partial^c f_1(x) = \begin{cases} 2x - 1, & x > 0, \\ [-2, -1], & x = 0, \\ 2x - 2, & x < 0, \end{cases}$$

$$\text{and } \ \partial^c f_2(x) = \begin{cases} 2x + e^x, & x > 0, \\ [-1, 1], & x = 0, \\ 2x - e^{-x}, & x < 0. \end{cases}$$

It is not hard to verify that $\tilde{x} = 0$ is an efficient minimizer of order 2 with $c = (1, 1)$. But $\tilde{x} = 0$ is not a solution of (GSVVLIP), as for any $y > 0$, there does not exist any ζ_i, $i = 1, 2$, such that

$$(\langle \zeta_1, \eta(y, \tilde{x}) \rangle, \langle \zeta_2, \eta(y, \tilde{x}) \rangle) \notin -\mathbb{R}^2_+ \setminus \{\mathbf{0}\}.$$

From Theorem 2 and 3, we have the following result.

Corollary 1: Let Γ be an invex set with respect to η such that η is skew and let each $f_i,\ i \in \mathcal{I}$ satisfies the Condition A, η satisfies the Condition C. If $\tilde{x} \in \Gamma$ solves (GSVVLIP), then $\tilde{x}$ is a solution of (GMVVLIP).

Now, we consider the following weak formulation of generalized Minty and Stampacchia vector variational-like inequality problems from Al-Homidan & Ansari (2010), in terms of Clarke subdifferential:

(WGMVVLIP) A weak generalized Minty vector variational-like inequality problem is to find $\tilde{x} \in \Gamma$, such that, for each $y \in \Gamma$ and for all $\xi_i \in \partial^c f_i(y), i \in \mathcal{I}$, we have

$$\langle \xi, \eta(y, \tilde{x})\rangle_p = (\langle \xi_1, \eta(y, \tilde{x})\rangle, ..., \langle \xi_p, \eta(y, \tilde{x})\rangle) \notin -\text{int}\mathbb{R}^p_+.$$

(WGSVVLIP) A weak generalized Stampacchia vector variational-like inequality problem is to find $\tilde{x} \in \Gamma$, such that, for each $y \in \Gamma$, there exists $\zeta_i \in \partial^c f_i(\tilde{x}), i \in \mathcal{I}$, we have

$$\langle \zeta, \eta(y, \tilde{x})\rangle_p = (\langle \zeta_1, \eta(y, \tilde{x})\rangle, ..., \langle \zeta_p, \eta(y, \tilde{x})\rangle) \notin -\text{int}\mathbb{R}^p_+.$$

IV. Relationship between (WGMVVLIP), (WGSVVLIP) and (NVOP)

In this section, we establish some results which show the relationship among the solutions of weak generalized Minty and Stampacchia vector variational-like inequality problems (WGMVVLIP), (WGSVVLIP) and strict minimizer of order m with respect to η of the nonsmooth vector optimization problem (NVOP).

Proposition 1: Let Γ be an invex set with respect to η such that η is skew and let each $f_i, i \in \mathcal{I}$ be strongly invex of order m with respect to η. If $\tilde{x}$ solves (WGSVVLIP), then $\tilde{x}$ is a solution of (WGMVVLIP).

Proof: Since $\tilde{x} \in \Gamma$ solves (WGSVVLIP), then for any $z \in \Gamma$, there exist $\zeta_i \in \partial^c f_i(\tilde{x}),\ i \in \mathcal{I}$, such that

$$(\langle \zeta_1, \eta(z, \tilde{x}), \ldots, \langle \zeta_p, \eta(z, \tilde{x})) \notin -\text{int}\mathbb{R}^p_+. \quad (28)$$

Since, $f_i,\ i \in \mathcal{I}$ is strongly invex of order m with respect to η, then from Lemma 1, $\partial^c f_i$ is strongly invariant monotone of order m with respect to η. Therefore, there exists a constant $c \in \text{int}\mathbb{R}^p_+$, such that for all $\zeta_i \in \partial^c f_i(\tilde{x})$, $\xi_i \in \partial^c f_i(z)$, and $z \in \Gamma$, we get

$$\langle \xi_i - \zeta_i, \eta(z, \tilde{x})\rangle \geq c_i[||\eta(z, \tilde{x})||^m + ||\eta(\tilde{x}, z)||^m], \quad \forall i \in \mathcal{I}. \quad (29)$$

From Eq. (28) and Eq. (29), we get

$$(\langle \xi_1, \eta(z, \tilde{x})\rangle, ..., \langle \xi_p, \eta(z, \tilde{x})\rangle) \notin -\text{int}\mathbb{R}^p_+.$$

Hence, $\tilde{x} \in \Gamma$ is a solution of (WGMVVLIP). ■

Al-Homidan & Ansari (2010) prove the following result.

Proposition 2: Let Γ be an invex set with respect to η and let each $f_i,\ i \in \mathcal{I}$ be locally Lipschitz. If $\tilde{x} \in \Gamma$ solves (WGMVVLIP), then $\tilde{x}$ solves (WGSVVLIP).

From Propositions 1 and 2, we have the following result.

Theorem 4: Let Γ be an invex set with respect to η such that η is skew and let each $f_i,\ i \in \mathcal{I}$ be strongly invex of order m. Then $\tilde{x}$ solves (WGSVVLIP) if and only if $\tilde{x}$ solves (WGMVVLIP).

Proposition 3: Let Γ be an invex set with respect to η such that η is skew and let each $f_i,\ i \in \mathcal{I}$ be strongly invex of order m. If $\tilde{x} \in \Gamma$ solves (WGSVVLIP), then $\tilde{x}$ is a strict minimizer of order m with respect to η of (NVOP).

Proof: Let $\tilde{x} \in \Gamma$ solves (WGSVVLIP), but not a strict minimizer of order m with respect to η of (NVOP). Then there exist $z \in \Gamma$ such that for any $c \in \text{int}\mathbb{R}^p_+$, we have

$$(f_1(z) - f_1(\tilde{x}) - c_1||\eta(z, \tilde{x})||^m, \ldots, f_p(z) - f_p(\tilde{x}) - c_p||\eta(z, \tilde{x})||^m) \in -\text{int}\mathbb{R}^p_+. \quad (30)$$

Since each $f_i, i \in \mathcal{I}$ is strongly invex of order m with respect to η, there exists a constant $c \in \text{int}\mathbb{R}^p_+$, such that

$$\langle \zeta_i, \eta(z, \tilde{x})\rangle \leq f_i(z) - f_i(\tilde{x}) - c_i||\eta(z, \tilde{x})||^m, \quad \forall \zeta_i \in \partial^c f_i(\tilde{x}). \quad (31)$$

From Eq. (30) and Eq. (31), for all $\zeta_i \in \partial^c f_i(\tilde{x}),\ i \in \mathcal{I}$, we get

$$(\langle \zeta_1, \eta(z, \tilde{x})\rangle, ..., \langle \zeta_p, \eta(z, \tilde{x})\rangle) \in -\text{int}\mathbb{R}^p_+,$$

which is contrary to our assumption that $\tilde{x}$ solves (WGSVVLIP). ■

Proposition 4: Let Γ be an invex set with respect to η such that η is skew and let each $f_i,\ i \in \mathcal{I}$ be strongly invex of order m. If $\tilde{x} \in \Gamma$ is a strict minimizer of order m with respect to η of (NVOP), then $\tilde{x}$ solves (WGMVVLIP).

Proof: Let $\tilde{x} \in \Gamma$ is a strict minimizer of order m with respect to η of (NVOP) but does not solves (WGMVVLIP). Therefore, there exist $z \in \Gamma$ and $\xi_i \in \partial^c f_i(z),\ i \in \mathcal{I}$, such that

$$(\langle \xi_1, \eta(z, \tilde{x})\rangle, \ldots, \langle \xi_p, \eta(z, \tilde{x})\rangle) \in -\text{int}\mathbb{R}^p_+. \quad (32)$$

Since η is skew and each $f_i,\ i \in \mathcal{I}$ is strongly invex of order m with respect to η, there exists a constant $c \in \text{int}\mathbb{R}^p_+$, such that, for all $\xi_i \in \partial^c f_i(z)$, we get

$$f_i(z) - f_i(\tilde{x}) \leq \langle \xi_i, \eta(z, \tilde{x})\rangle - c_i||\eta(z, \tilde{x})||^m, \quad \forall i \in \mathcal{I}. \quad (33)$$

Since $c_i > 0$, $\forall i \in \mathcal{I}$, from Eq. (33), we have

$$f_i(z) - f_i(\tilde{x}) - c_i||\eta(z, \tilde{x})||^m \leq \langle \xi_i, \eta(z, \tilde{x})\rangle - 2c_i||\eta(z, \tilde{x})||^m, \leq \langle \xi_i, \eta(z, \tilde{x})\rangle. \quad (34)$$

From Eq. (32) and Eq. (34), we get

$$(f_1(z) - f_1(\tilde{x}) - c_1||\eta(\tilde{x}, z)||^m, ..., f_p(z) - f_p(\tilde{x}) - c_p||\eta(\tilde{x}, z)||^m) \in -\text{int}\mathbb{R}^p_+,$$

which contradicts the assumption that $\tilde{x}$ is an efficient minimizer of order m with respect to η of (NVOP). ■

From Theorem 4 and Propositions 3 and 4, we have the following result.

Theorem 5: Let Γ be an invex set with respect to η such that η is skew and let each $f_i,\ i \in \mathcal{I}$ be strongly invex of order m. Then $\tilde{x} \in \Gamma$ solves (WGSVVLIP) if and only if it is a strict minimizer of order m with respect to η of (NVOP).

V. Existence of solutions for (GMVVLIP) and (GSVVLIP)

In this section, by employing KKM-Fan theorem, we establish the conditions under which the solution of generalized Minty and Stampacchia vector variational-like inequality problems (GMVVLIP) and (GSVVLIP) exist.

The following definition and lemma are from Li & Yu (2017).

Definition 10: Let X be a nonempty subset of topological vector space Y. A multifunction $\Phi : X \to 2^Y$ is a KKM map if for any finite subset $\{y_1, y_2, \dots, y_n\}$ of X, it satisfies

$$co\{y_1, y_2, \dots, y_n\} \subset \bigcup_{i=1}^{n} \Phi(y_i),$$

where $co\{y_1, y_2, \dots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \dots, y_n\}$.

Lemma 3 (KKM-Fan theorem): Let X be a nonempty convex subset of a topological vector space Y, and let $\Phi : X \to 2^Y$ be a KKM map with closed values. If there is a point $\bar{x} \in X$ such that $\Phi(\bar{x})$ is compact, then

$$\bigcap_{x \in X} \Phi(x) \neq \phi.$$

The following theorem establishes the conditions for the existence of solutions for generalized Minty vector variational-like inequality problem (GMVVLIP).

Theorem 6: Let each $f_i : \Gamma \to \mathbb{R},\ i \in \mathcal{I}$ be locally Lipschitz and the following conditions are satisfied:

1) For each $i \in \mathcal{I},\ \langle \xi_i, \eta(x,y)) \rangle + \langle \zeta_i, \eta(y,x) \rangle \geq 0$, $\forall\, \xi_i \in \partial^c f_i(x)\, and\, \zeta_i \in \partial^c f_i(y)$.
2) η is affine in second argument.
3) For all $x \in \Gamma,\ (\langle \zeta_1, \eta(x,x)) \rangle, \dots, \langle \zeta_p, \eta(x,x) \rangle) \notin \mathbb{R}^p_+ \setminus \{\mathbf{0}\},\ \ \forall \zeta_i \in \partial^c f_i(x),\ i \in \mathcal{I}$.
4) The set valued map $G(x) = \{y \in \Gamma : (\langle \xi_1, \eta(x,y) \rangle, \dots, \langle \xi_p, \eta(x,y) \rangle) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\},\ \forall\, \xi_i \in \partial^c f_i(x), i \in \mathcal{I}\}$, $\forall x \in \Gamma$ is closed valued.
5) There exists nonempty compact sets $P, Q \subset \Gamma$ such that Q is convex and for each $y \in \Gamma \setminus P$, there exists $x \in Q$, such that $y \notin G(x)$.

Then (GMVVLIP) is solvable on Γ.

Proof: We define a map

$$H(x) := \{y \in \Gamma : (\langle \zeta_1, \eta(y,x) \rangle, \dots, \langle \zeta_p, \eta(y,x) \rangle) \notin \mathbb{R}^p_+ \setminus \{\mathbf{0}\}, \forall \zeta_i \in \partial^c f_i(y), i \in \mathcal{I}\},\ \forall x \in \Gamma.$$

From definition of H, it is clear that $x \in H(x)$. Therefore, $H(x)$ is nonempty. Now we have to show that $H(x)$ is a KKM map on Γ. On contrary, suppose that $H(x)$ is not a KKM map, then there exists $\{y_1, y_2, \dots y_n\} \subset \Gamma,\ \lambda_j \geq 0$, $j = 1, 2, \dots, n$, with $\sum_{j=1}^{n} \lambda_j = 1$, such that

$$\tilde{x} = \sum_{j=1}^{n} \lambda_j y_j \notin \bigcup_{j=1}^{n} H(y_j). \tag{35}$$

Hence, for all $y_j,\ j = 1, 2, ..., n$, we have

$$(\langle \zeta_1, \eta(\tilde{x}, y_j) \rangle, \dots, \langle \zeta_p, \eta(\tilde{x}, y_j) \rangle) \in \mathbb{R}^p_+ \setminus \{\mathbf{0}\},\quad \forall \zeta_i \in \partial^c f_i(\tilde{x}),\ i \in \mathcal{I}. \tag{36}$$

that is, for each $y_j,\ j = 1, 2, ..., n$ and for all $\zeta_i \in \partial^c f_i(\tilde{x})$, we have

$$\langle \zeta_i, \eta(\tilde{x}, y_j) \rangle \geq 0,\ \forall i \in \mathcal{I}, \text{ with strict inequality for atleast one } i. \tag{37}$$

Multiplying Eq. (37) by $\lambda_j,\ j = 1, 2, ..., n$ and adding the resulting inequalities, we get

$$\sum_{j=1}^{n} \lambda_j \langle \zeta_i, \eta(\tilde{x}, y_j) \rangle \geq 0, \text{ with strict inequality for atleast one } i. \tag{38}$$

Since η is affine in second argument, from Eq. (38), we have

$$\langle \zeta_i, \eta(\tilde{x}, \sum_{j=1}^{n} \lambda_j y_j) \rangle \geq 0.$$

From the definition of $\tilde{x}$, we get

$$\langle \zeta_i, \eta(\tilde{x}, \tilde{x}) \rangle \geq 0\ \ \forall i \in \mathcal{I}, \text{ with strict inequality for atleast one } i.$$

It follows that,

$$(\langle \zeta_1, \eta(\tilde{x}, \tilde{x}) \rangle, ..., \langle \zeta_p, \eta(\tilde{x}, \tilde{x}) \rangle) \in \mathbb{R}^p_+ \setminus \{\mathbf{0}\},\quad \forall \zeta_i \in \partial^c f_i(\tilde{x}),\ i \in \mathcal{I},$$

which contradicts our assumption. Hence, $H(x)$ is a KKM map on Γ. Now, we have to show that $H(x) \subset G(x),\ \forall x \in \Gamma$. If $\tilde{x} \notin G(x)$, then there exist $x \in \Gamma$, such that

$$(\langle \xi_1, \eta(x, \tilde{x}) \rangle, \dots, \langle \xi_p, \eta(x, \tilde{x}) \rangle) \in -\mathbb{R}^p_+ \setminus \{\mathbf{0}\},\quad \forall \xi_i \in \partial^c f_i(x),\ i \in \mathcal{I}. \tag{39}$$

From Eq. (39), it follows that,

$$\langle \xi_i, \eta(x, \tilde{x}) \rangle \leq 0,\ \forall i \in \mathcal{I}, \text{ with strict inequality for atleast one } i. \tag{40}$$

From condition (i), we have

$$\langle \xi_i, \eta(x, \tilde{x}) \rangle + \langle \zeta_i, \eta(\tilde{x}, x) \rangle \geq 0,\ \ \xi_i \in \partial^c f_i(x),\quad \zeta_i \in \partial^c f_i(\tilde{x}),\ i \in \mathcal{I}. \tag{41}$$

From Eq. (40) and Eq. (41), we get

$$\langle \zeta_i, \eta(\tilde{x}, x) \rangle \geq 0,\ \ \forall i \in \mathcal{I}, \text{ with strict inequality for atleast one } i. \tag{42}$$

From Eq. (42), there exist $x \in \Gamma$, such that

$$(\langle \zeta_1, \eta(\tilde{x}, x) \rangle, ..., \langle \zeta_p, \eta(\tilde{x}, x) \rangle) \in \mathbb{R}^p_+ \setminus \{\mathbf{0}\},\ \ \forall \zeta_i \in \partial^c f_i(\tilde{x}).$$

Therefore, $\tilde{x} \notin H(x)$. Since, we have shown that $H(x) \subset G(x)$ for any $x \in \Gamma$. Hence, G is also a KKM map. From conditions 4 and 5, $G(x)$ is a closed subset of the compact set. Thus, $H(x)$ is also a compact set. From the KKM-Fan Theorem

$$\bigcap_{x \in \Gamma} G(x) \neq \phi,$$

which means that for all $x \in \Gamma$, we get a $\tilde{x} \in \Gamma$, such that

$$(\langle \xi_1, \eta(x, \tilde{x}) \rangle, ..., \langle \xi_p, \eta(x, \tilde{x}) \rangle) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\},\quad \forall \xi_i \in \partial^c f(x),\ i \in \mathcal{I}.$$

Hence, (GMVVLIP) is solvable on Γ. ■

The following example illustrates the significance of Theorem 6.

Example 2: Let $f = (f_1, f_2) : [-1,1] \to \mathbb{R}^2$ and $\eta : [-1,1] \times [-1,1] \to \mathbb{R}^2$ be defined as:

$$f_1(x) = \begin{cases} x^2+1, & x \geq 0, \\ x^2 + e^{-x}, & x < 0, \end{cases} \quad f_2(x) = \begin{cases} x^2, & x \geq 0, \\ x^2 - x, & x < 0, \end{cases}$$

$$\text{and } \eta(x,y) = \begin{cases} 1-y, & x \geq 0 \text{ and } y < 0, \\ x-y, & \text{elsewhere.} \end{cases}$$

It is clear that η is affine in second argument. Now, we can evaluate that

$$\partial^c f_1(x) = \begin{cases} 2x, & x > 0, \\ [-1,0], & x = 0, \\ 2x - e^{-x}, & x < 0 \end{cases}$$

$$\text{and } \partial^c f_2(x) = \begin{cases} 2x, & x > 0, \\ [-1,0], & x = 0, \\ 2x-1, & x < 0. \end{cases}$$

Now, we show that the conditions (1)-(5) of the Theorem 6 are satisfied.

(i) We have to verify the condition (1), that is, for all $\xi_i \in \partial^c f_i(x)\, and\, \zeta_i \in \partial^c f_i(y)$,

$$\langle \xi_i, \eta(x,y)) \rangle + \langle \zeta_i, \eta(y,x) \rangle \geq 0, \quad i = 1,2.$$

For the function f_1, the following cases arise

Case (i) $x > 0,\, y > 0$,

$$\langle 2x, x-y \rangle + \langle 2y, y-x \rangle = 2(x-y)^2 \geq 0.$$

Case (ii) $x < 0,\, y < 0$,

$$\begin{aligned} &\langle 2x - e^{-x}, x-y \rangle + \langle 2y - e^{-y}, y-x \rangle \\ &= 2(x-y)^2 - (e^{-x} - e^{-y})(x-y) \\ &\geq 0. \end{aligned}$$

Case (iii) $x > 0,\, y < 0$,

$$\begin{aligned} &\langle 2x, 1-y \rangle + \langle 2y - e^{-y}, y-x \rangle \\ &= 2y^2 - 4xy + (x-y)e^{-y} + 2x \\ &\geq 0. \end{aligned}$$

Case (iv) $x < 0,\, y > 0$,

$$\begin{aligned} &\langle 2x - e^{-x}, x-y \rangle + \langle 2y, 1-x \rangle \\ &= 2x^2 - 4xy + 2y + (y-x)e^{-x} \\ &\geq 0. \end{aligned}$$

Similarly, we can show that the condition (1) is also satisfied for the function f_2.

(ii) Since, for all $x \in [-1,1]$, $\eta(x,x) = 0$, therefore,

$$(\langle \xi_1, \eta(x,x) \rangle, \langle \xi_2, \eta(x,x) \rangle) \notin \mathbb{R}^2_+ \setminus \{\mathbf{0}\}, \quad \forall \xi_i \in \partial^c f_i(x),\, i = 1,2.$$

(iii) It is clear from the definition of G,

$$G(x) = \begin{cases} [-1,x], & x > 0, \\ [0,1], & x = 0, \\ [x,1], & x < 0. \end{cases}$$

(iv) Let us consider the set $P = [-1,0]$ and $Q = [0,1]$. It is clear that Q is convex and for all $y \in [-1,1] \setminus P$, there exists a $x < y$, such that $y \notin G(x)$.

Furthermore, we can verify that $\tilde{x} = 0$ is a solution of (GMVVLIP).

On the lines of the proof of Theorem 6, we have the following theorem for the existence of solution for (GSVVLIP).

Theorem 7: Let each $f_i : \Gamma \to \mathbb{R},\ i \in \mathcal{I}$ be locally Lipschitz of order m and the following conditions are satisfied:

1) For each $i \in \mathcal{I},\ \langle \xi_i, \eta(y,x)) \rangle + \langle \zeta_i, \eta(x,y) \rangle \geq 0$, $\forall \xi_i \in \partial^c f_i(x)$ and $\zeta_i \in \partial^c f_i(y)$.
2) η is affine in second argument.
3) For all $x \in \Gamma,\ (\langle \xi_1, \eta(x,x) \rangle, \ldots, \langle \xi_p, \eta(x,x) \rangle) \notin \mathbb{R}^p_+ \setminus \{\mathbf{0}\},\ \forall \xi_i \in \partial^c f_i(x)$.
4) The set valued map $G(x) = \{y \in \Gamma : (\langle \zeta_1, \eta(x,y) \rangle, \ldots, \langle \zeta_p, \eta(x,y) \rangle) \notin -\mathbb{R}^p_+ \setminus \{\mathbf{0}\}, \forall x \in \Gamma, \zeta_i \in \partial^c f_i(y)\}$ is closed valued.
5) There exists nonempty compact sets $P, Q \subset \Gamma$ such that Q is convex and for each $y \in \Gamma \setminus P$, there exists $x \in Q$, such that $y \notin G(x)$.

Then (GSVVLIP) is solvable on Γ.

VI. Conclusion and future directions

In this paper, we have established the relationship between the solutions of generalized Minty and Stampacchia vector variational-like inequality problems (GMVVLIP), (GSVVLIP) and efficient minimizers of order m of nonsmooth vector optimization problems (NVOP) using the assumption of the strongly invex function of order m. We also consider the weak formulation of generalized Minty and Stampacchia vector variational-like inequality problems (WGMVVLIP), (WGSVVLIP) and establish the relationship between the solutions of these vector variational-like inequality problems and strict minimizers of order m of the nonsmooth vector optimization problem (NVOP). Employing KKM-Fan theorem, we establish the existence result for the solutions of generalized Minty and Stampacchia vector variational-like inequality problems (GMVVLIP), (GSVVLIP). Suitable examples are given to illustrate the significance of these results. The results of the paper extend and unify some earlier results of Bhatia (2008), Li & Yu (2017) and Upadhyay et al. (2017) to the nondifferentiable case as well as for a more general class of nonconvex functions. Further, the tools of Michel-Penot subdifferentials, Michel & Penot (1984), Mordukhovich limiting subdifferentials, Mordukhovich (2006) or convexificators, Demyanov & Jeyakumar (1997) may be employed to sharpen the corresponding results in this paper.

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