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Stability Analysis of a Complex Dynamics of a SIR Epidemic Model with Bilinear Incidence Rate and Treatment

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Abstract: **In this article, A SIR epidemic model with bilinear incidence rate has been proposed and the existing threshold requirements of all classifications of equilibrium points are obtained. Further, we study the global and local stability of the disease-free and endemic equilibriums of the model. An optimal control problem is formed and solved. Some numerical simulations works are carried out to demonstrate our results. In this process, our results generalized and improved any results in existing literature.**

Index Terms: **SIR epidemic model, basic reproduction number, optimal treatment control, Global Stability**

I. INTRODUCTION

Evolution of medical science made a diagnosis of deadly diseases facile and helped to deal and curb them more efficiently than it was a few decades ago. Nonetheless, infectious diseases, which can spread either by direct exposure or through any other channel, do have the potential to become epidemic and stretch to all corners of the earth. Furthermore, varieties of new infectious diseases, fatal and seemingly convoluted composition, are coming into existence along with the time, and are needed to be controlled by different tools. Moreover, such tools needed to be developed to check such diseases efficaciously. Hence apart from medical field specialists pioneers, experts and researchers from diversified fields are now partaking in to find an optimal way to check widespread of such diseases and if possible exterminate them as well. Fortunately, a valuable tool Mathematical modelling which can come in handy to analyze the nature of infectious diseases and can help to develop control methodology to deal with them, Daniel Bernoulli (1760) formulated the first epidemiological model for smallpox.

But then came a cessation in the field of epidemiological modelling, which got its breakthrough when pioneer (Hamer, 1906) put forward the outcome of his research in the early twentieth century. Down the line, in 1927, Kermack and Mckendrick presented a preeminent deterministic compartment model in order to analyze the outbreak of Black Death in London in (1665-1666) and Mumbai plague havoc (1906). The results of that model were precise in predicting the nature of outbreaks and were in unison with recorded data of epidemics.

Recently, many mathematical model has been proposed to control the effect of infectious diseases. (See Eckalbar et al., 2011; Hosono et al., 1995 ; Hu et al., 2012; Jana et al.,2016; Wang ,2006; Wang , 2012; Zhou et al.,2012)

The most critical task is to sort out the form of the disease-communication an infected person to a healthy person. The mass action form is the most common form of disease transmission; this form pretends that the number of a healthy person who will get affected is proportional to the number of those who will contact the infected beings.

Researchers like Arino et al., 2004; Buonomo et al., 2008; Makinde, 2007; Thomasey et al., 2011 have used only vaccination in a theoretical study of human epidemiology while researcher like Hu et al., 2012; Qiu et al., 2009 have used treatment control only. Further some researcher like Kar et al., 2013; Laarabi et al., 2015; Okosun et al., 2011; Tchuenche et al. 2011 have used both controls in their study. In this research, treatment control only has been used as a control to introduce a model of an epidemic system. Further, this would be relevant that other infectious diseases, such as influenza, tuberculosis measles and many others flues, are generally overcome by suitable treatment control Feng et al., 1995; Hyman et al., 1998; Wang, 2006; Wu et al., 2000. The treatment function is mainly taken in the linear form Kar et al., 2013; Laarabi et al., 2015 etc. In 2006, researcher Wang (2006) put forward a limited resource epidemic model for treatment.

Zhang and Liu (2008) and Eckalbar and Eckalbar (2011) implemented non-linear treatment function in their researches. It

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was considered in their study that numbers of the recuperated patient depend on two factors, first being treatment control, and another is a number of an infected person in the system. Hence it's plausible to suppose treatment function $T(\mu,I)=f(\mu I)$. In Zhang et al., 2008 proposed saturated type treatment function $T(I)$ of the form $T(I)=\frac{rI}{1+\alpha I}, r>0, \alpha>0$, to measure the effectiveness of the infected being delayed for treatment. When the number of infected persons are inadequate and tends to a fixed limit for a tremendous value of I, Zang and Liu's saturated treatment function have been proved to be advantageous to produce non-linear treatment outcome.

Moreover, every feasible value of I has finite and a continuous value in this treatment function. Likewise Jana et al., 2015 considered treatment function as a function of both infectious human (I) and the control (μ), and it is considered as $T(I)=\frac{r\mu I}{1+b\mu I}, r>0, \alpha>0$. It was observed that for the very small value of I or μ, the treatment function also tends to zero, and for a value of I, which is very high, it tends to a finite limit. Such treatment function has reflected natural epidemic model, and hence this treatment function is utilized in this suggested model.

II. Manuscript Organization

The organization of this manuscript as follows. First section is devoted to the introduction and literature review of the proposed problem. In the Section III, we present the model diagram and formulation of the mathematical model while Section IV contains the analysis of the model for fixed control with and we derived local and global stability. Section V proposed the application of optimal control diseases. In Section VI, we construct some numerical examples for the optimal control Problem. Finally, we give the concluding remarks in conclusion.

III. Mathematical Model formation

This section presents the SIR epidemic model with bilinear incidence rate and introduces some correlative definitions of differential and algebraic systems where the population is divided into three compartments, namely, the susceptible population $S(t)$, infected population $I(t)$, and recovered population $R(t)$. It is supposed that the parasites of the diseases are transferred to susceptible individuals through direct contact with infected individuals.

Jana (2015) considered the SIR epidemic model with saturated incidence rate and a continually differentiable treatment function with the saturation phenomenon of the limited medical resources were studied as follows:

$$\frac{dS}{dt}=A-\frac{\beta SI}{1+\alpha I}-dS+pR$$
$$\frac{dI}{dt}=\frac{\beta SI}{1+\alpha I}-\frac{ruI}{1+buI}-dI-\delta I-mI$$
$$\frac{dR}{dt}=\frac{ruI}{1+buI}-dR-pR+mI$$

Inspiring by this work, we consider a SIR epidemic model with the bilinear incidence rate αSI and a continually differentiable treatment function $T(I)=\frac{r\mu I}{1+b\mu I}$ formulated by Wu et al., 2000.The model can be described as follows:

$$\left.\begin{aligned}\frac{dS}{dt}&=a-\alpha SI-d_1S+\beta R\\ \frac{dI}{dt}&=\alpha SI-\frac{r\mu I}{1+b\mu I}-d_1I-\delta_0 I-mI\\ \frac{dR}{dt}&=\frac{r\mu I}{1+b\mu I}-d_1R-\beta R+mI\end{aligned}\right\} \quad (1)$$

where, β is the rate of losing immunity at time t, a is the recruitment rate (including births and immigrations) of susceptible individuals at any time t, and all the new recruited populations go to the susceptible class, $\alpha(\geq 0)$ is the disease transmission rate and bilinear transmission function αSI, r is positive quantity, b is non negative quantity and μ be the treatment control parameter, d_1 is the natural death rate of population, δ_0 related to disease induced death rate. Historically, a portion of almost every infectious disease recovers naturally, so the portion mI of infected individuals moves to improved (recovered) class due to their natural recovery. All parameters β, a, b, d_1, r α and δ_0 are positive.

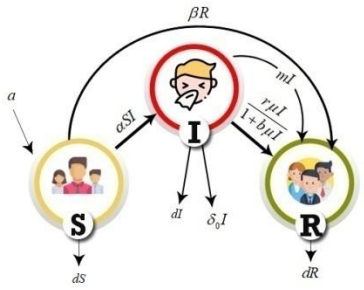

Diagram of SIR Model

IV. Stability Analysis of the Model

Here, a study is presented by the dynamical behavior of the system (1), when the treatment control parameter is taken as fixed. Various complex phenomena like basic reproduction number, uniform boundedness, the existence of all feasible equilibrium and stability criteria, etc. are completely explained in this section.

A. Finiteness of the system

Let us study the boundedness characteristic of the total population in the model:

"Theorem 1 the solutions of the model (1) are uniformly bounded."

Then we get,

$$\frac{dN}{dt} = a - d_1 N - \delta_0 I ,$$

i.e. $$\frac{dN}{dt} + d_1 N \le a$$

Now, by integrating both sides of the above inequality and using the theory of the differential inequality presented by (Birkhoff et al., 1982), we have

$$0 < N(S,I,R) \le \frac{a}{d_1}(1 - e^{-d_1 t}) + N(S(0),I(0),R(0))e^{-d_1 t} ,$$

By using limit, $t \to \infty$, we get $0 < N \le \frac{a}{d_1}$,

Therefore all the solution of (1) that initiating at $\{\Re_+^3 \setminus 0\}$ are circumscribed in the region

$$\Re = \left\{ (S,I,R) \in \Re_+^3 : N = \frac{a}{d_1} + \xi \right\}. \qquad (2)$$

For any $\xi > 0$ and for $t \to \infty$ .Consequently, the theorem proved.

As from (2), it can be said that all the solutions of the system (1) are positive and uniformly bounded, it is claimed that there exists a constant $\mu_1 > 0$ such that $\mu_1 < min\{liminf\ S(t), liminf\ I(t), liminf\ R(t)\}$.

B. Basic Reproduction Number

The basic reproduction number is R_0 products of infection rates and durations of infection. In other words, the basic reproduction number R_0 is defined as the number of infected individuals caused by a single infective individual considered by Van et al., 2002.

$$R_0 = \frac{\alpha a}{d_1(d_1 + \delta_0 + m + r\mu)}$$

C. Equilibrium Points and their Feasibility Criteria

The system (1) always has the diseases free equilibrium $E_0(\frac{a}{d_1}, 0, 0)$ for any set parameters. The model (1) has two other possible equilibria where the disease remains in the order. These two equilibria are known as endemic equilibria with coordinates $E_1(S_1, I_1, R_1)$ and $E_2(S_2, I_2, R_2)$. Further, the conditions of the existence of endemic equilibrium will be found at $E_2(S^*, I^*, R^*)$

An endemic equilibrium always satisfies

$$\frac{dS}{dt} = a - \alpha SI - d_1 S + \beta R = 0$$

$$\frac{dI}{dt} = \alpha SI - \frac{r\mu I}{1 + b\mu I} - d_1 I - \delta_0 I - mI = 0$$

$$\frac{dR}{dt} = \frac{r\mu I}{1 + b\mu I} - d_1 R - \beta R + mI = 0$$

By simple calculation, we get $S^* = \frac{1}{\alpha}\left[\frac{r\mu}{1 + b\mu I} + (d_1 + \delta_0 + m)\right]$, $R^* = \frac{(r\mu + m + bm\mu I)}{(d_1 + \beta)(1 + b\mu I)}$ and I^* are roots of following quadratic equation

$A_1 I^{*2} + A_2 I^* + A_3 = 0$, such that $I_2 \ge I_1$, (3)

with

$A_1 = \alpha\beta bm\mu$

$A_2 = (d_1 + \beta)\{\alpha ab\mu - b\mu(d_1 + \delta_0 + m)(\alpha + d_1)\} + \alpha\beta(r\mu + m)$

$A_3 = (d_1 + \beta)\{\alpha a - \alpha r\mu - d_1 r\mu - (d_1 + \delta_0 + m)(\alpha + d_1)\}$

Now,

$$I^* = \frac{-[(d_1 + \beta)\{\alpha ab\mu - b\mu(d_1 + \delta_0 + m)(\alpha + d_1)\} + \alpha\beta(r\mu + m) \pm \sqrt{\Delta}}{2\alpha\beta bm\mu}$$

where discriminant of equation (3) is

$$\Delta = [(d_1 + \beta)\{\alpha ab\mu - b\mu(d_1 + \delta_0 + m)(\alpha + d_1)\} + \alpha\beta(r\mu + m)]^2 - \frac{4\alpha\beta bm\mu(d_1 + \beta)}{(d_1 + \delta_0 + m + r\mu)}\left\{\frac{\alpha a}{d_1(d_1 + \delta_0 + m + r\mu)} - 1\right\}$$

"Lemma 1 Existence criteria of endemic equilibrium(s),

I. If either of μ or b is zero then equation (3) enhances a linear equation on I and system have a unique endemic equilibrium with $I = \frac{A_3}{A_2}$ which is attainable if and only if $R_0 > 1$.

II. If both μ and b are non-zero, then (3) is a quadratic equation and this equation may have two positive solutions for I if $A_2 < 0$ and $R_0 < 1$. Once again if $A_2^2 \ge 4A_1A_3$ then it has two positive solutions. In this situation, there are two positive equilibria say $E_1(S_1, I_1, R_1)$ and $E_2(S_2, I_2, R_2)$.

III. Finally, if both μ and b are non zero and $R_0 < 1$, then the quadratic equation (3) has only one change of sign and therefore by using Descarte's rule of sign it can be required that the system has a unique feasible equilibrium."

D. Local Asymptotic Stability

Here, the local stability of the equilibria will be examined by using the Routh-Hurwitz criterion and examining the eigenvalues of the Jacobian matrices of the model (1) at the equilibria

"Theorem 2 The diseases free equilibrium E_0 is locally asymptotically stable when $R_0 < 1$ and is unstable when $R_0 > 1$."

Proof The Jacobian matrix of system (1) at $E_0(\frac{a}{d_1},0,0)$ is

$$J(E_0)=\begin{pmatrix} -d_1 & -\frac{\alpha a}{d_1} & \beta \\ 0 & \frac{\alpha a}{d_1}-(d_1+\delta_0+m+r\mu) & 0 \\ 0 & r\mu & -(d_1+\beta) \end{pmatrix}=0 \quad (4)$$

The characteristics equation of system (4) at E_0 is the following form:

$$(d_1+\lambda)(\lambda^2+C_1\lambda+C_2)=0 \quad (5)$$

where $C_1 = 2d_1+\beta+\delta_0+m+r\mu-\frac{\alpha a}{d_1}$ and

$$C_2=(d_1+\beta)\left(d_1+\delta_0+m+r\mu-\frac{\alpha a}{d_1}\right)$$
$$=(d_1+\beta)(d_1+\delta_0+m+r\mu)(1-R_0)$$

Clearly, $\lambda_1=-d_1$ is the eternally negative root of (5), and all other roots of (5) are solved through $(\lambda^2+C_1\lambda+C_2)=0$, which has negative roots if and only if $(d_1+\delta_0+m+r\mu-\frac{\alpha a}{d_1})>0$ i.e. $R_0=\frac{\alpha a}{d_1(d_1+\delta_0+m+r\mu)})<1$. So the diseases free equilibrium E_0 is locally asymptotically stable when $R_0<1$ and is unstable when $R_0>1$.

Note 1: In this study, the control parameter is one of the vital parameters of the system parameters. Consequently from the above theorem, we may assume that the diseases free equilibrium is locally asymptotically stable if $\mu > \frac{(\alpha a - d_1^2 - d_1 m - d_1\delta_0)}{d_1 r}$

"Theorem 3 The endemic equilibrium $E_2(S^*,I^*,R^*)$ is locally asymptotically stable, if $R_0>1$."

Proof The Jacobian matrix of system (1) at endemic equilibrium $E_2(S^*,I^*,R^*)$ is denoted by $J(E^*)$, and defined by

$$J(E^*)=\begin{pmatrix} -d_1-\alpha I^* & -\alpha S^* & \beta \\ \alpha I^* & \alpha S^*-\frac{r\mu}{(1+b\mu I^*)}+\frac{br\mu^2 I^*}{(1+b\mu I^*)^2}-(d_1+\delta_0+m) & 0 \\ 0 & \frac{r\mu}{(1+b\mu I^*)}-\frac{br\mu^2 I^*}{(1+b\mu I^*)^2} & -(d_1+\beta) \end{pmatrix}$$

Consider $V_1=\alpha I^*$, $V_2=\alpha S^*$ and $V_3=\frac{r\mu}{(1+b\mu I^*)}+\frac{br\mu^2 I^*}{(1+b\mu I^*)^2}$, then Jacobian matrix is given by

$$J(E^*)=\begin{pmatrix} -d_1-V_1 & -V_2 & \beta \\ V_1 & V_2-V_3-(d_1+\delta_0+m) & 0 \\ 0 & V_3 & -(d_1+\beta) \end{pmatrix}$$

The characteristics equation of system (3) at E_2 is given by

$$\begin{vmatrix} -d_1-V_1-\lambda_1 & -V_2 & \beta \\ V_1 & V_2-V_3-(d_1+\delta_0+m)-\lambda_1 & 0 \\ 0 & V_3 & -(d_1+\beta)-\lambda_1 \end{vmatrix}=0$$

$$(d_1+V_1+\lambda_1)(d_1+\beta+\lambda_1)\{V_2-V_3-(d_1+\delta_0+m)-\lambda_1\}$$
$$-V_1V_2(d_1+\beta+\lambda_1)+\beta V_1V_3=0$$

Putting $K_1=d_1+V$, $K_2=d_1+\beta$ and $K_3=d_1+\delta_0+m$ in above equation and solving, we get

$\lambda_1^3+D_1\lambda_1^2+D_2\lambda_1+D_3=0$.

where

$D_1=K_1+K_2+K_3+V_3-V_2$,

$D_2=K_1K_2+K_2K_3+K_3K_1+K_1V_3+K_2V_3-K_1V_2-K_2V_2+V_1V_2$,

$D_3=K_1K_2V_3+K_1K_2K_3-V_1V_2K_2-K_1K_2V_2-\beta V_1V_2$,

$D_1D_2-D_3=K_1^2(V_3-V_2)+(K_1+K_2)(K_2+K_3)(V_3-V_2)+$

$(V_3-V_2)^2(K_1+K_2)+(K_1+K_2+K_3)(K_1K_2+K_2K_3+K_3K_1)$

$+(K_2K_3+K_3K_1)(V_3-V_2)+V_2K_2d_1+\beta V_1V_3$.

Clearly, $D_1,D_2,D_3>0$ and $D_1D_2-D_3>0$, provided $V_3>V_2$ In such a situation, the Routh-Hurwitz criterion conceding determines whether each root has negative real parts and confirms the system's stability without solving the characteristic equation itself. Consequently, by the Routh-Hurwitz criterion, it ensures that the endemic equilibrium $E_2(S^*,I^*,R^*)$ is locally asymptotically stable. This ascertains the proof.

Note 2: It is discerned that the disease-free equilibrium claims from stable to unstable as R_0 increases to 1. So, it may be determined that at $R_0=1$, the system passes by a bifurcation approximately its disease free equilibrium E_0

Note 3: For $R_0<1$, E_* becomes unstable. Again for $R_0<1$, the disease-free equilibrium is locally asymptotically stable in the region $\Re$, and hence there exists only one stable equilibrium of the model. Consequently, the diseases will be exterminated from the system.

E. Global Stability

The global stability of the endemic steady states is analyzed in this section. To examine the global asymptotic stability of $E_2(S^*,I^*,R^*)$, the method of Li et al., 1996, the geometric approach method involving the global stability of an endemic equilibrium is applied. Furthermore, the adequate conditions for which the endemic equilibrium is globally asymptotically stable are obtained. The geometric approach method is illustrated succinctly. Suppose

$$x' = \Omega(x) \qquad (6)$$

Where $\Omega : D_1 \to \Re^n$, $D_1 \subset \Re^n$ is open set and is clearly connected, and $x \in D_1$, $x \mapsto \Omega''(x) \in \Re^n$, $\Omega(x) \epsilon C^1(D_1)$.

Let x^* be a equilibrium of (6). Now, it is recollected that x^* is called to be globally stable in D_1 if it is locally stable and all trajectories in D_1 converge to x^*. Consider that the following hypotheses hold.

(Δ_1) $\exists$ A compact absorbing set $\kappa_1 \subset D_1$.

(Δ_2) Equation (6) has a sole equilibrium x^* in D_1.

The basic idea of this method is that if the equilibrium x^* is locally stable, then the stability is ensured given that (Δ_1) and (Δ_2) holds and no non-constant periodic solution of (6) exists. Hence, sufficient conditions on Ω capable of preventing the presence of such solutions have to be identified. If (Δ_1) and (Δ_2) satisfy and equation (6) hold Bendixson criterion that is robust under C^1 local ϵ-perturbations of Ω at all non-equilibrium non-wandering points for (6), and then x^* is globally stable and robust under C^1 local ϵ –perturbation, explained by Li and Muldowney. Let $P(x)$ be a $\binom{n}{2} \times \binom{n}{2}$ matrix valued function, that is, C^1 on D and also suppose that

$$B = P_\Omega P^{-1} + P \frac{\partial \Omega^{[2]}}{\partial x} P^{-1} \qquad (7)$$

where P_Ω is

$$\frac{\partial P_{ij}^*}{\partial x}\Omega = \left.\frac{\partial P_{ij}}{\partial t}\right|_{(6)} \qquad (8)$$

and the matrix $J^{[2]}$ is the another additive compound matrix of the Jacobian matrix J, that is, $J(x) = D_1\Omega(x)$. In general, for a $n \times n$ matrix $J = (J_{ij})$, $J^{[2]}$ is a $\binom{n}{2} \times \binom{n}{2}$ matrix and in the particular case $n = 3$ one has,

$$J^{[2]} = \begin{pmatrix} J_{11} + J_{22} & J_{23} & -J_{13} \\ J_{32} & J_{11} + J_{33} & J_{12} \\ -J_{31} & J_{21} & J_{22} + J_{33} \end{pmatrix}$$

Let us assumed that the *Lozinskiĭ* measure μ of B with respect to a vector norm $\|.\|$ in $\Re^n$, $N = \binom{n}{2}$ expressed by

$$\mu(B) = \lim_{h \to 0^+} \frac{\|I + hB\| - 1}{h}$$

It is shown in (Li et al., 1996) that if (Δ_1) and (Δ_2) satisfy, condition

$$q = \lim_{t \to \infty} \sup \sup_{x_0 \in K} \frac{1}{t} \int_0^t \mu(B(x(s, x_0)))ds < 0 \qquad (9)$$

There are no orbits giving rise to a simple closed rectifiable curve in D_1 which is invariant for (6) that is, periodic orbits, homoclinic orbits, and heteroclinic cycles, it is proved in [18]. In particular, condition (9) is showed to be a robust Bendixson criterion for (6). Moreover, it is mentioned that, under the premises (Δ_1) and (Δ_2), further condition (9) indicates the local stability of x^*. Besides, it is robust under C^1 local perturbations of Ω near any non equilibrium point that is non-wandering. In particular, the following global-stability outcome is shown in Li et al., 1996.

"Lemma.2 Considered that D_1 is simply connected and that the premises (Δ_1) and (Δ_2) hold. Then the unique equilibrium x^* of (6) is globally stable in D_1 if $q < 0$. This result is shown in Li et al., 1996."

Here, The global stability analyzed and obtained of the endemic equilibrium $E_2(S^*, I^*, R^*)$.

"Theorem 4 If $R_0 > 1$, then the endemic equilibrium $E_2(S^*, I^*, R^*)$ of the model (1) is globally stable"

Proof The second additive matrix of the model (1) is expressed by $J^{[2]}(S, I, R)$, where Jocabian of the model (1) is expressed by $J(E^*)$ and defined by

$$J(E^*) = \begin{pmatrix} -d_1 - \alpha I & -\alpha S & \beta \\ \alpha I & \alpha S - \frac{r\mu}{(1 + b\mu I)} + \frac{br\mu^2 I}{(1 + b\mu I)^2} - (d_1 + \delta_0 + m) & 0 \\ 0 & \frac{r\mu}{(1 + b\mu I)} - \frac{br\mu^2 I}{(1 + b\mu I)^2} & -(d_1 + \beta) \end{pmatrix}$$

$$J^{[2]} = \begin{pmatrix} \alpha(S - I) - \frac{r\mu}{(1 + b\mu I)} + \frac{br\mu^2 I}{(1 + b\mu I)^2} - (2d_1 + \delta_0 + m) & 0 & -\beta \\ \frac{r\mu}{(1 + b\mu I)} - \frac{br\mu^2 I}{(1 + b\mu I)^2} & -\alpha I - 2d_1 - \beta & -\alpha S \\ 0 & \alpha I & \alpha S - \frac{r\mu}{(1 + b\mu I)} + \frac{br\mu^2 I}{(1 + b\mu I)^2} - (2d_1 + \delta_0 + m + \beta) \end{pmatrix}$$

Choose the function $P = P(S, I, R) = diag(\frac{S}{I}, \frac{S}{I}, \frac{S}{I})$; it understand that

$P^{-1} = diag(\frac{I}{s}, \frac{I}{s}, \frac{I}{s})$, and

$$P_\Omega = diag\left\{\frac{S'}{I} - \frac{S}{I^2}I', \frac{S'}{I} - \frac{S}{I^2}I', \frac{S'}{I} - \frac{S}{I^2}I'\right\},$$

$P_\Omega P^{-1} = diag\left\{\frac{S'}{S}-\frac{I'}{I},\frac{S'}{S}-\frac{I'}{I},\frac{S'}{S}-\frac{I'}{I}\right\}$, and

$PJ^{[2]}P^{-1} = J^{[2]}$.

It is defined that,

$B_0 = P_\Omega P^{-1} + PJ^{[2]}P^{-1} = B_0 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where

$$B_{11} = \alpha(S-I) - \frac{r\mu}{(1+b\mu I)} + \frac{br\mu^2 I}{(1+b\mu I)^2} - (2d_1+\delta_0+m) + \frac{S'}{S} - \frac{I'}{I},$$

$$B_{12} = \beta,$$

$$B_{21} = \frac{r\mu}{(1+b\mu I)} - \frac{br\mu^2 I}{(1+b\mu I)^2},$$

$$B_{22} = \begin{pmatrix} \frac{S'}{S}-\frac{I'}{I}-\alpha I-2d_1-\beta & -\alpha S \\ \alpha I & \frac{S'}{S}-\frac{I'}{I}+\alpha S-\frac{r\mu}{(1+b\mu I)}+ \\ & \frac{br\mu^2 I}{(1+b\mu I)^2}-(2d_1+\delta_0+m+\beta) \end{pmatrix}$$

Let (u,v,w) be vector in $\Re^3$; its norm $\|.\|$ is represented as $\|(u,v,w)\| = max\{|u|,|v|+|w|\}$.

Also, let $\gamma(B_0)$ be the *Lozinskiĭ* measure with respect to this norm. We choose

$\gamma(B_0) \le sup\{g_1, g_2\}$,

$= sup\{\gamma_1(B_{11})+|B_{12}|, \gamma_1(B_{22})+|B_{21}|\}$.

where $|B_{21}|$ are matrix norm with respect to L^1 vector norm and γ_1 is the *Lozinskiĭ* measure with respect to L^1 norm, then

$$\gamma_1(B_{11}) = \frac{S'}{S}-\frac{I'}{I}+\alpha(S-I)-\frac{r\mu}{(1+b\mu I)}+\frac{br\mu^2 I}{(1+b\mu I)^2} - (2d_1+\delta_0+m),$$

and

$$\gamma_1(B_{22}) = \frac{S'}{S}-\frac{I'}{I}-\alpha I-2d-\beta+ max\left\{-\alpha I, \alpha S-\frac{r\mu}{(1+b\mu I)^2}-\delta_0-m\right\},$$

Then we get

$g_1 = \frac{S'}{S}-\frac{I'}{I}+\alpha(S-I)-\frac{r\mu}{(1+b\mu I)^2}-2d_1-\delta_0-m+\beta)$, and

$$g_2 = \frac{S'}{S}-\frac{I'}{I}+\frac{r\mu}{(1+b\mu I)^2}-2d_1-\beta+ max\left\{-\alpha I, \alpha S-\frac{r\mu}{(1+b\mu I)^2}-\delta_0-m\right\}$$

From equation second of system (1), we have

$\frac{I'}{I} = \alpha S - \frac{r\mu}{(1+b\mu I)} - (d_1+\delta_0+m)$, therefore it can be written

$g_1 = \frac{S'}{S}-\alpha I-d_1+\beta+\frac{br\mu^2 I}{(1+b\mu I)^2}$, and

$$g_2 = \gamma_1(B_{22})+|B_{21}| = \frac{S'}{S}-d_1-\beta-\alpha S+\delta_0+m+\frac{r\mu}{(1+b\mu I)^2} + \frac{r\mu}{(1+b\mu I)}$$

$\gamma(B_0) \le sup\{g_1, g_2\}$

$$= sup\left\{\begin{matrix} \frac{S'}{S}-\alpha I-d_1+\beta+\frac{br\mu^2 I}{(1+b\mu I)^2}, \frac{S'}{S}-d_1-\beta-\alpha S+\delta_0+ \\ m+\frac{r\mu}{(1+b\mu I)^2}+\frac{r\mu}{(1+b\mu I)} \end{matrix}\right\}$$

$$\gamma(B_0) \le \frac{S'}{S}-d_1+max\left\{\begin{matrix} \beta-\alpha I+\frac{br\mu^2 I}{(1+b\mu I)^2}, -\beta-\alpha S+\delta_0+m \\ +\frac{r\mu}{(1+b\mu I)^2}+\frac{r\mu}{(1+b\mu I)} \end{matrix}\right\}$$

$$\gamma(B_0) \le \frac{S'}{S}-(d_1-\omega)$$

where $\omega = max\left\{\begin{matrix} \beta-\alpha I+\frac{br\mu^2 I}{(1+b\mu I)^2}, -\beta-\alpha S+\delta_0+m+ \\ \frac{r\mu}{(1+b\mu I)^2}+\frac{r\mu}{(1+b\mu I)} \end{matrix}\right\}$

By integrating both sides simultaneously, it is obtained that

$\frac{1}{t}\int_0^t \gamma(B_0)ds \le \frac{1}{t}ln\frac{S(t)}{S(0)} - (d_1-\omega)$,

$\lim_{t\to\infty} sup\, sup\, \frac{1}{t}\int_0^t \gamma(B_0)ds < -(d_1-\omega)$,

$\lim_{t\to\infty} sup\, sup\, \frac{1}{t}\int_0^t \gamma(B_0)ds \le 0$.

Consequently, by the outcome of Li et al., 1996, it shows that $E_2(S^*, I^*, R^*)$ is globally asymptotically stable.

The proof is complete by Lemma 2.

Note 4: By using the preceding theorem, it can be assumed that if the natural death rate of each population class is more significant than some threshold parametric condition then $E_2(S^*, I^*, R^*)$ may be globally asymptotically stable.

F. Sensitivity of basic Reproduction Number

In an epidemic model, the basic reproduction number R_0 is an essential parameter. It is difficult to control diseases if R_0 is extensive. In the absence of treatment control, consider R_0^* to be

the basic reproduction number. The relationship between these two reproduction numbers is described in the next theorem.

"Theorem 5 We always have $R_0 \le R_0^*$."

Proof By definition of R_0^* which is given above, it is easy to obtain that R_0^*

$$\frac{\alpha a}{d_1(d_1+\delta_0+m)} \ge \frac{\alpha a}{d_1(d_1+\delta_0+m+r\mu)} = R_0$$

Note 5: It is easy to state that the treatment control would always provide a reduced basic reproduction number from the above theorem. Therefore, proper treatment control still provides some better outcomes compared to no control case.

V. Application of Optimal Control Diseases

Treatment control can be used as a constant, but initially, it should be time conditional. From this, it can be concluded that this control must be applied according to the necessity. The cost of control must always be considered, but in the individual case, the control should be applied as much as possible, when diseases expect to be epidemic. When the main objective is to use the treatment control according to the necessity of time, it cannot always be neglected because of the high amount of treatment cost. Consequently, an optimal control problem is formed and determined with the help of the subsidiary equation.

A. Optimal Control Problem Formulation

The initial purpose of analyzing infectious diseases is trying to exterminate the infection from the community and to improve control. By using this purpose, an optimal control problem is constructed with a concurrent motive to overcome infected people as much as possible.

The objective function to be optimized is formulated as follows:

$$J_0 = \min_{\mu} \int_0^{t_1} (\rho_1 I + \rho_2 \mu) dt, \qquad (10)$$

Subject to differential equations (1)

A linear function is defined as in terms of available infected individual I and cost of treatment control μ, which is studied in (10), thus we suppose J_0 as linear function of both I and μ.

Now here, the condition is to explore a control μ^* as follows:

$$J_0(\mu^*) = \min_{\mu \in \Theta} J_1(\mu), \qquad (11)$$

Here, a set for the control is defined as $\Theta = \{\mu : is\ measurable\ and\ 0 \le \mu(t) \le 1\ for \in [0,1]\}$. The control parameters consistently required is $0 \le \mu(t) \le 1$ Makinde (2007). $\mu(t)$ gets its highest value 1 when treatment is provided to all people, while when none of the infected population get treatment, $\mu(t)$ consider its minimal value 0. In intermediate conditions $\mu(t)$ refers $(0,1)$ open interval.

We create the Lagrangian of the problem, to explain the framework, which is provided by

$$L(I,\mu) = \rho_1 I + \rho_2 \mu \qquad (12)$$

Now, the Hamiltonian H for our problem is constructed as given below

$$H(S,I,R,\mu,\lambda_1,\lambda_2,\lambda_3) = L(I,\mu) + \lambda_1(t)\frac{dS}{dt} + \lambda_2(t)\frac{dI}{dt} + \lambda_3(t)\frac{dR}{dt} \qquad (13)$$

where $\lambda_i(t)$ for $i = 1,2,3$ are co-state variables or adjoint variables and can be concluded by solving the subsequent system of ordinary differential equations:

$$\dot{\lambda}_1(t) = \frac{dH}{dS} = \lambda_1(t)(\alpha I + d_1) - \lambda_2(t)\alpha I, \qquad (14)$$

Again

$$\dot{\lambda}_2(t) = \frac{dH}{dI}$$

$$= -A_1 + \lambda_1(t)\alpha S - \lambda_2(t)\left[\alpha S - \frac{r\mu}{(1+b\mu I)^2} - (d_1+\delta_0+m)\right]$$

$$-\lambda_3(t)\left[\frac{r\mu}{(1+b\mu I)^2} + m\right], \qquad (15)$$

$$\dot{\lambda}_3(t) = \frac{dH}{dR} = -\lambda_1(t)\beta + \lambda_3(t)(d_1+p). \qquad (16)$$

Satisfying transversality conditions:

$$\dot{\lambda}_i(t_1) = 0, i = 1,2,3 \qquad (17)$$

Consider optimum value of S,I,R are given by $\bar{S},\bar{I},\bar{R}$ and assume $\{\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3\}$ be solution of system (14) to (16). Then following theorem is given:

"Theorem 6 There is an optimal control $\mu^*(t)$ such that

$$J_1(I(t),\mu^*(t)) = \min_{\mu} J_1(I(t),\mu(t))$$

Subject to the differential equations (14) to (16)".

Proof In the given theorem, variable $\mu(t)$ is convex as all the co state variables are non negative Furthermore, the control space is closed and convex. Hence the optimal control is bounded, and we consider the existence of an optimal control $\mu^*(t)$. Therefore the theorem applying Pontryagin's Maximum Principle (1962), the subsequent theorem, is now asserted and proved.

"Theorem 7 The optimal control μ^* which minimizes J_1 over the region Θ is provided by

$$\mu^* = max\{0, min(\bar{\mu}, 1)\}.$$

Where $\bar{\mu} = \frac{1}{b\bar{I}}\left(\sqrt{\frac{r(\bar{\lambda}_2 - \bar{\lambda}_3)\bar{I}}{\rho_2}} - 1\right)$".

Proof Applying the optimality condition $\frac{\delta H}{\delta \mu} = 0$, we have $\bar{\mu} = \frac{1}{b\bar{I}}\left(\sqrt{\frac{r(\bar{\lambda}_2 - \bar{\lambda}_3)\bar{I}}{\rho_2}} - 1\right)$.

Over the control is bounded with lower and upper bound is 0 and 1, respectively. i.e. $\mu < 0$ then $\bar{\mu} = 0$, $\mu > 1$ then $\bar{\mu} = 1$, and otherwise $\bar{\mu} = \mu$. Thus for the control $\bar{\mu} = \mu^*$, the optimal value of the function J_1 is obtained, consequently proved theorem.

VI. Numerical Simulations

To see dynamical behavior of system (1), we present some numerical simulations by using MATLAB in support of the theoretical investigation given in previous sections.

Case-I: The dynamical behavior of system (1), choosing the parameters as under

$\alpha = 0.01$, $a = 100$, $d_1 = 0.2$, $b = 0.5$, $\beta = 0.4$, $\delta_0 = 0.05$, $m = 5.8$ $r = 0.78$ and $\mu = 0.5$, next by computing, we obtained basic reproduction number $R_0 = 0.7763 < 1$ and model (1) has a disease-free equilibrium $E_0\left(\frac{a}{d_1}, 0, 0\right) = (500, 0, 0)$. In this case, $S(t)$ approaches to its steady state value while $I(t)$ and $R(t)$ approaches to zero as time goes to infinity. Thus the disease disappears and dies out. The numerical simulation is manifested in Figure 1. It observes that E_0 is globally asymptotically stable. Figure 1(a)-1(c) shows the dynamic behaviors of system (1).

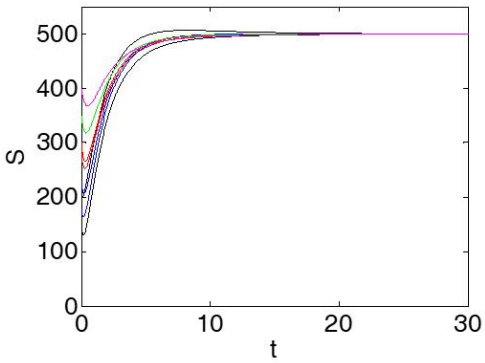

Figure: 1(a) Population behavior

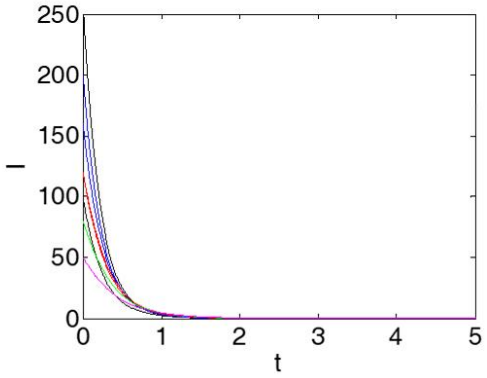

Figure: 1(b) Population behavior

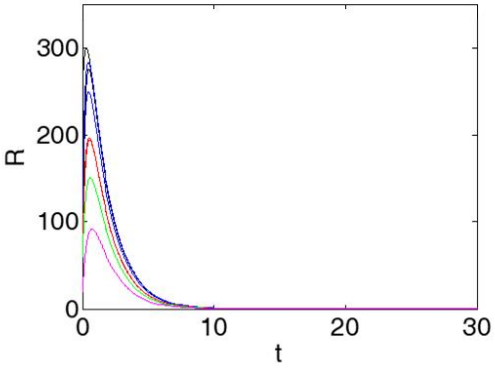

Figure: 1(c) Population behavior

Figure1: The figures represent that the diseases dies out

Case-II: The dynamical behavior of system (1), choosing the parameters as under

$\alpha = 0.01$, $a = 500$, $d_1 = 0.2$, $b = 0.5$, $\beta = 0.4$, $\delta_0 = 0.3$, $m = 0.1$ $r = 2$ and $\mu = 0.8$. Next, by computing, the basic reproduction number $R_0 = 11.364 > 1$, and model (1) has a endemic equilibrium $E_2(S^*, I^*, R^*) = (60.021, 1816.659, 309.434)$ is obtained, for the above choice of parameter, it is seen that $S(t)$, $I(t)$ and $R(t)$ approach to their steady-state values as time goes to infinity and the diseases becomes endemic (See figure 2(a)-2(c)).

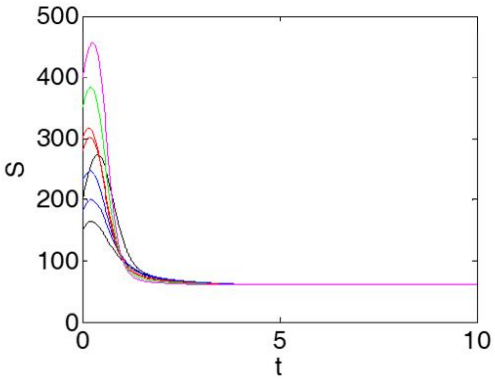

Figure: 2(a) Population behavior

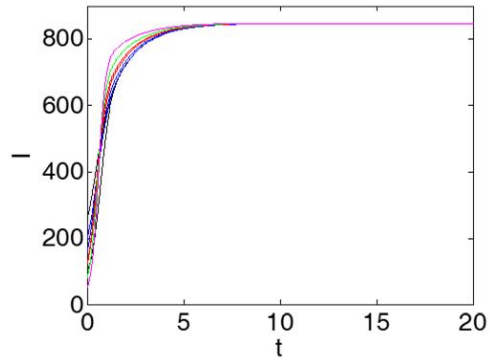

Figure: 2(b) Population behavior

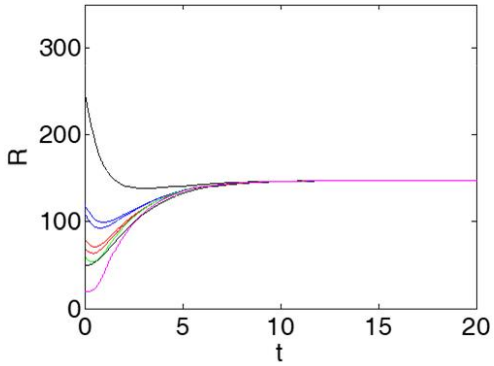

Figure: 2(c) Population behavior

Figure2: The figures represent that the diseases dies endemic

Case III: it is observed that from the system (14)-(16), towards the final time the value of λ_1, λ_2, λ_3 are all goes to zero. These phenomena assure that to get the minimum value of the objective function (10), as the changes of the state variables S, I and R, the rate of change of Hamiltonian H will be increasing. (Figure 3(a)-3(b)).

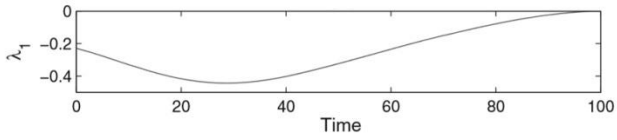

Figure: 3(a)

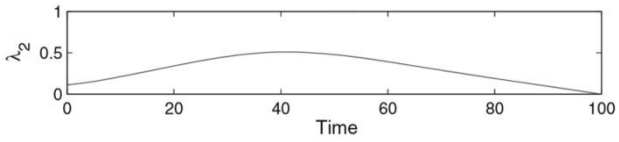

Figure: 3(b)

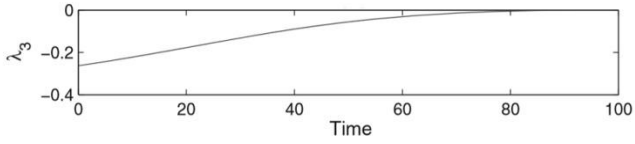

Figure: 3(c)

Figure 3: Time evolution of the adjoint variables λ_1, λ_2, λ_3

CONCLUSION

A compartmental epidemic model has been analyzed by considering susceptible, infected, and recovered through treatment control method. Moreover, available treatment control and infected individuals, both are considered as independent variables for treatment function. The complex-dynamics of the systemic model has been discussed through different facets, local and global asymptotic stability of the system has also been studied. The stability of the equilibrium states has also been carried out using some of the tested parameters from literature reviewed in this paper.

Diseases, which are transmissible through human contact and which could be cured through effective medical treatment are the subject of this study. From analysis, this could be maintained that by reasonably higher treatment control, the endemic equilibrium turns unsteady. Hence, a stable endemic equilibrium could lose stability when treatment controls value is increased.

The main subject of the study is to control the disease via proper use of treatment control. For this purpose, the optimal control technique would be an excellent tool. Here, the optimal control technique has been used to minimize the number of infected individuals and the total cost associated with the treatment control.

REFERENCES

Arino J., Cooke K. L., Van D. P. & Velasco H. J. (2004). An epidemiology model that includes a leaky vaccine with a general waning function. Dynamical System Series B, 4(2), 479–495.

Bernoulli D. (1760). A new analysis of the mortality caused by the Smallpox, and advantages of the inoculation to prevent, Mathematical Physics Academy Royal Sciences, Paris, 1, 1-45.

Birkhoff G. & Rota G. C. (1982). Ordinary differential equations, Ginn, Boston.

Buonomo B., D'Onofrio A. & Lacitignola D. (2008) Global stability of an SIR epidemic model with information dependent vaccination, Mathematical Biosciences, 216.

Eckalbar J. C. & Eckalbar W. L. (2011). Dynamics of an epidemic model with quadratic treatment, Nonlinear Analysis: Real World Applications, 12(1), 320–332.

Feng Z. & Thieme H. R.(1995). Recurrent outbreaks of childhood diseases revisited: The impact of isolation, Mathematical Biosciences, 128(2), 93–130.

Hamer W. H. (1906). Epidemics disease in England, The Lancet, 1, 733-739.

Hosono Y. & Ilyas B. (1995). Traveling waves for a simple diffusive epidemic model, Math Model Method Apply Science, 5, 935–966.

Hu Z., Ma W. & Ruan S. (2012). Analysis of SIR epidemic models with nonlinear incidence rate and treatment, Mathematical Biosciences, 238, 12–20.

Hyman J. M. & Li J. (1998). Modeling the effectiveness of isolation strategies in preventing STD epidemics, SIAM J Applied Mathematics, 58, 912–925.

Jana S., Haldar P. & Kar T. K. (2016). Complex dynamics of an epidemic model with vaccination and treatment controls, International Journal of Dynamics and Control, 4(3), 318-329

Jana S. & Kar T. K. (2013). A mathematical study of a prey–predator model in relevance to pest control, Nonlinear Dynamics, 74(3), 667–683.

Jana S., Nandi S. & Kar T. K. (2015). Complex dynamics of an SIR epidemic model with saturated incidence rate and treatment, International Journal of Applied Computational Mathematics, Springer Science, 9, 65-84.

Kar T. K. & Jana S. (2013). A theoretical study on mathematical modelling of an infectious disease with application of optimal control. Bio systems, 111, 37–50.

Kar T. K. & Jana S. (2013). Application of three controls optimally in a vector-borne disease mathematical study. Communications in Nonlinear Science and Numerical Simulation, 18(10), 2868–2884.

Kermack W. O. & McKendrick A. G. (1927). A contribution to the mathematical theory of epidemics, Proceedings of the Royal Society London, 115, 700–721.

Laarabi H., Abta A. & Hattaf K. (2015). Optimal control of a delayed SIRS epidemic model with vaccination and treatment. Acta Biotheoretica, 63(2), 87-97.

Li M.Y. & Muldowney J. S. (1996). A geometric approach to global stability problem, SIAM Journal on Mathematical Analysis , 27(4), 1070–1083.

Makinde O. D. (2007). A domain decomposition approach to a SIR epidemic model with constant vaccination strategy, Applied Mathematics and Computation, 184(2), 842–848.

Okosun K. O., Ouifki R. & Marcus N. (2011). Optimal control analysis of a Malaria disease transmission model that includes treatment and vaccination with waning immunity, Bio Systems, 106, 136–145.

Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V. & Mishchenko E.F. (1962). The mathematical theory of optimal processes. Wiley, New York.

Qiu Z. & Feng Z. (2009). Transmission dynamics of influenza model with vaccination and antiviral treatment, Bulletin of Mathematical Biology, 72(1), 1–33.

Tchuenche J. M., Khamis S. A., Agusto F.B. & Mpeshe S. C. (2011). Optimal control and sensitivity analysis of an influenza model with treatment and vaccination. Acta Biotheoretica, 59(1), 1–28.

Thomasey D. H. & Martcheva M. (2011). Serotype replacement of vertically transmitted diseases through perfect vaccination, Journal of Biological System, 16(2), 255–277.

Van den Driessche P. & Watmough J. (2002). Reproduction numbers and sub-threshold endemic equilibria for compartmental models of diseases transmission, Mathematical Biosciences, 180, 29–48.

Wang W. (2006). Backward bifurcation of an epidemic model with treatment, Mathematical Biosciences, 201(2), 58–71.

Wang W., Cai Y., Wu M., Wang K. & Li Z. (2012). Complex dynamics of a reaction–diffusion epidemic model, Nonlinear Analysis Real World Applications, 13(5), 2240–2258.

Wu L. & Feng Z. (2000). Homoclinic bifurcation in an SIQR model for childhood diseases, Journal of Differential Equations, 168, 150–167

Zhang X. & Liu, X. (2008). Backward bifurcation of an epidemic model with saturated treatment, Journal of Mathematical Analysis and Applications, 348(1), 433–443.

Zhou L. & Fan M. (2012). Dynamics of an SIR epidemic model with limited medical resources revisited, Nonlinear Analysis: Real World Applications, 13(1), 312–324.

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