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# A New Extension of Lindley Distribution And Its Application

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***Abstract*—In statistical literature, various lifetime distributions have been proposed for analysing the lifetime data. Lindley distribution is one of them. It is a one-parameter model. But its suitability is restricted to the data having an increasing failure rate. In many real situations, data may possess other shapes of hazard rate function like- decreasing, bathtub, or up-sided down bathtub, etc. In this research article, we propose a generalization of Lindley distribution which is capable to fit a variety of datasets having different shapes of hazard rate function. Several statistical characteristics and properties of this distribution are also studied. Finally, to show the suitability and applicability of the proposed model in real scenarios two different datasets have been considered.**

***Index Terms*—Lifetime distribution, Lindley distribution, hazard rate function, parameters, Maximum Likelihood Estimation.**

## I. Introduction

Hazard rate or failure rate is an important key for modeling and analysing data in the area of medical, engineering, finance, insurance, and others. In statistical literature, the exponential distribution is the most exploited lifetime distribution for lifetime data having a constant hazard rate function. But, if the data shows a non-constant hazard rate function, the use of exponential distribution may lead to wrong results. To overcome this difficulty, several distributions have been developed such as Weibull, gamma, log-normal, Lindley distribution, and many others. Weibull and gamma distributions contain two parameters and their hazard rate function have increasing hazard (IHR) and decreasing hazard (DHR) shapes while Lindley distribution proposed by Lindley, D. V. (1958) has single parameter and shape of its hazard rate function is increasing.

Latter, Ghitany, M. E., Atieh, B., & Nadarajah, S. (2008) has discussed the application of Lindley distribution in real life scenario and become a very popular lifetime model. After that versions generalizations of the Lindley distribution have been discussed by various authors, few of them are Ghitany, M. E. & Al-Mutairi, D. K. (2008), Mazucheli, J. & Achcar, J. A. (2011), Nadarajah, S., Bakouch, H. S., & Tahmasbi, R. (2011), Ghitany (M. E.), Pararai, M., Warahena-Liyanage, G., & Oluyede, B. O. (2015), Sharma, V. K., Singh, S. K., Singh, U., & Agiwal, V. (2015), Zeghdoudi, H. & Nedjar, S. (2016), Asgharzadeh (A.), Maurya, S. K., Kaushik, A., Singh, S. K., & Singh, U. (2017b) and Maurya, S. K., Singh, S. K., & Singh, U. (2020) etc.

There are several generalization techniques available in statistical literature, in order to get flexible distributions. For example, Lehmann, E. L. (1953), Kumaraswamy, P. (1980), Gupta, R. C., Gupta, P. L., & Gupta, R. D. (1998), Shaw, W. & Buckley, I. (2007), Cordeiro, G. M. & De Castro, M. (2011), Cordeiro, G. M., Ortega, E. E. M., & Daniel, C. C. D. C. (2013), Gomes, A. E., Da-Silva, C. Q., & Cordeiro, G. M. (2015), Kumar, D., Singh, U., & Singh, S. K. (2015), Maurya, S. K., Kaushik, A., Singh, R. K., Singh, S. K., & Singh, U. (2016), Mahdavi, A. & Kundu, D. (2017), Aryal (G. R. & Yousof), Dey, S., Nassar, M., and Kumar, D. (2017), Maurya, S., Kaushik, A., Singh, S., & Singh, U. (2017a), Maurya, S. K., Kumar, D., Singh, S. K., & Singh, U. (2018) and Goyal, T., Rai, P. K., & Maurya, S. K. (2019) etc.

Here, we proposed a new generalization of Lindley distribution by using the same technique suggested by Cordeiro, G. M. & De Castro, M. (2011), which added two additional shape parameters to the chosen baseline distribution. If $G(x)$ be the CDF of some baseline model, then the CDF $F(x)$ of new distribution is

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$$F(x) = \left[1 - (1 - G(x))^{\alpha}\right]^{\beta}; \quad x > 0, \quad \alpha,\ \beta > 0. \tag{1}$$

For $\alpha = 1$, whatever $\beta$ may be, it reduces to exponentiated type distribution defined by Gupta, R. C., Gupta, P. L., & Gupta, R. D. (1998) and for $\alpha = \beta = 1$, it reduces to the baseline distribution. This shows that this generalization method may show greater flexibility in terms of fitting criterion.
Let $f(x)$ be the PDF corresponding to CDF $F(x)$ given in equation (1), then

$$f(x) = \alpha\beta\left[1 - G(x)\right]^{\alpha-1}\left[1 - (1 - G(x))^{\alpha}\right]^{\beta-1} g(x); \quad x > 0, \quad \alpha,\ \beta > 0. \tag{2}$$

Let $G(x)$ be the CDF of Lindley distribution, then

$$G(x) = 1 - e^{-\theta x}\left[1 + \frac{\theta x}{(1+\theta)}\right] \quad ; x > 0, \quad \theta > 0. \tag{3}$$

Using equation (3) in equations (1) and (2), we get the CDF & PDF of the new distribution as follows

$$F(x) = \left[1 - \left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta} \tag{4}$$

and

$$f(x) = \frac{\alpha\beta\theta^2}{(1+\theta)}(1+x)e^{-\theta x}\left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha-1} \times \left[1 - \left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta-1}; \quad x > 0,\ \alpha,\ \beta,\ \theta > 0. \tag{5}$$

There is a nice physical interpretation of the proposed generalization technique; see Cordeiro, G. M. & De Castro, M. (2011) for more details.

The rest of the paper is organized in a given sequence. In Section II, we discussed some statistical characteristics of proposed distribution such as reliability, hazard rate, nature of its distribution. Section III, deals with some statistical properties like moments, conditional moments, mean deviation about mean and median, quartile function, moment generating function, cumulative generating function, order statistic, and its probability distribution function, and entropy of the proposed model. Section IV, we discussed methods of estimation of parameters of the proposed model.

In Section V, two different datasets are taken to check the applicability of the proposed distribution to the real problems related to the medical field, and its performance is also compared with two other existing distributions, and the conclusion is presented in Section VI.

## II. Statistical Characteristics

In this section, we have derived various statistical characteristics of the proposed distribution such as reliability function, hazard rate function and shape of PDF and CDF.

### A. Reliability Function

The reliability function $R(x)$, is the probability that the system will not fail before time $x$ is obtained for proposed distribution as

$$R(x) = P(X \geq x) = 1 - \left[1 - \left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta}. \tag{6}$$

### B. Shapes of the PDF, CDF and hazard rate function of the proposed distribution

The shapes of PDF & CDF reflect the idea whether the distribution is symmetric or skewed. For different combination of the values of the parameters $\beta,\ \alpha$ and $\theta$, we have plotted CDF (4) and PDF (5) of the proposed of distribution in Fig. II-B. This figure shows that the proposed distribution exhibits a right-skewed type of distribution. Next, the hazard rate function is an important tool of lifetime data analysis. For our proposed model, the same is obtained as follows

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\alpha\beta\theta^2}{(1+\theta)}(1+x)e^{-\theta x}\left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha-1} \times \frac{\left[1 - \left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta-1}}{1 - \left[1 - \left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta}} \tag{7}$$

The proposed model is flexible in terms of hazard rate function. The Fig. 2 shows that the nature of the hazard rate function of the proposed distribution has non-monotone (bathtub) and monotone (IHR and DHR) shaped.

## III. Statistical Properties of the Proposed Distribution

### A. Moments

As we know that the soul of a human body is vested in the heart, in the same way, the soul of a given distribution is characterized by its moments, with the help of these one can identify the nature of the considered distribution. If $X$ be a random variable follows the proposed model, then the $r^{th}$ moment about origin of $X$ is $\frac{\alpha\beta\theta^2}{1+\theta}K(\beta,\alpha,\theta,r,\delta)$. Where $K(\beta,\alpha,\theta,r,\delta)$ is defined below.

***Lemma* 3.1**

$$K(\beta,\alpha,\theta,r,\delta) = \int_0^{\infty} x^r(1+x)e^{-\delta x}\left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha-1} \times \left[1 - \left(e^{-\theta x}\left(1 + \frac{\theta x}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta-1} dx$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j+1} C_i^{\beta-1} C_j^{\alpha i+\alpha-1} C_k^{j+1} \times \frac{(-1)^i\theta^j}{(1+\theta)^{\alpha i+\alpha-1}} \frac{\Gamma(k+r+1)}{[\theta(\alpha i+\alpha-1)+\delta]^{k+r+1}}$$

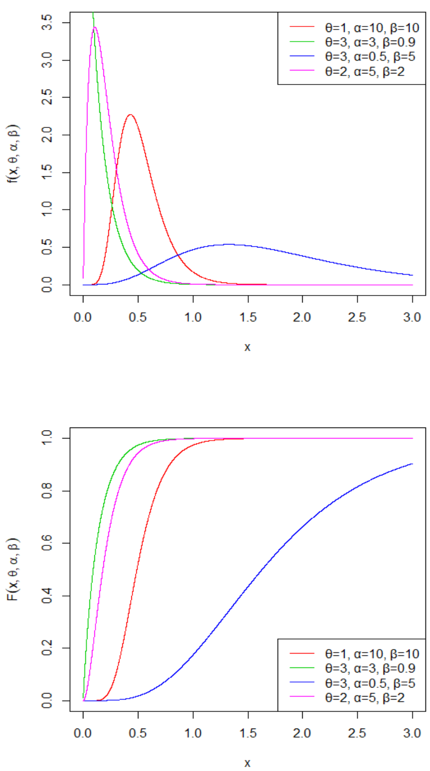


Fig. 1. Plots of probability density function and cumulative distribution function

***proof* :**

$K(\beta,\alpha,\theta,r,\delta)$

$$= \sum_{i=0}^{\infty}(-1)^i C_i^{\beta-1}\int_0^{\infty} x^r(1+x)e^{-\delta x} \times \left(e^{-\theta x}\left(1+\frac{\theta x}{(1+\theta)}\right)\right)^{\alpha+i\alpha-1} dx$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i C_i^{\beta-1}}{(1+\theta)^{\alpha i+\alpha-1}}\int_0^{\infty}(1+\theta(1+x))^{\alpha i+\alpha-1}x^r \times (1+x)\exp\left[-\theta x(\alpha i+\alpha-1)-\delta x\right]dx$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i C_i^{\beta-1}}{(1+\theta)^{\alpha i+\alpha-1}}\sum_{j=0}^{\infty}C_j^{\alpha i+\alpha-1}\theta^j\int_0^{\infty}x^r \times (1+x)^{j+1}\exp[-x(\theta(\alpha i+\alpha-1)+\delta)]dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}C_i^{\beta-1}C_j^{\alpha i+\alpha-1}\frac{(-1)^i\theta^j}{(1+\theta)^{\alpha i+\alpha-1}}\sum_{k=0}^{j+1}C_k^{j+1} \times \int_0^{\infty}x^{k+r}\exp[-x(\theta(\alpha i+\alpha-1)+\delta)]dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j+1}C_i^{\beta-1}C_j^{\alpha i+\alpha-1}C_k^{j+1}\frac{(-1)^i\theta^j}{(1+\theta)^{\alpha i+\alpha-1}} \times \frac{\Gamma(k+r+1)}{[\theta(\alpha i+\alpha-1)+\delta]^{k+r+1}}$$

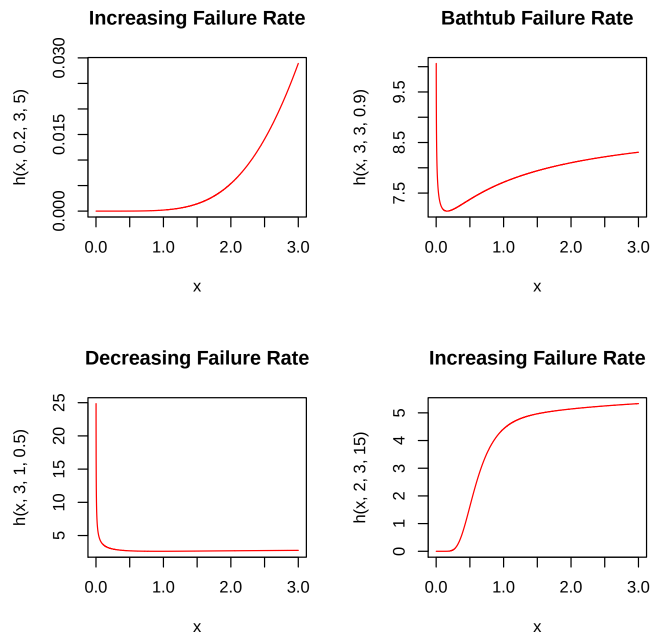


Fig. 2. Hazard rate function plot for various choice of parameters.

Thus, the $r^{th}$ raw moments about origin is given below

$$E[X^r]=\frac{\alpha\beta\theta^2}{(1+\theta)}K(\beta,\alpha,\theta,r,\delta)$$

and the first four moments about origin can be obtained $r=1,2,3$ and 4.

## *B. Conditional Moments*

For lifetime models, it is also of interest to know what the value of $E[X^n|X>t]$ (conditional moments) is, before its calculation for the proposed distribution, first we state the following lemma.

***Lemma 3.2***

$$L(\beta,\alpha,\theta,r,\delta,t)=\int_t^{\infty}x^r(1+x)e^{-\delta x} \times \left(e^{-\theta x}\left(1+\frac{\theta x}{(1+\theta)}\right)\right)^{\alpha-1} \times \left[1-\left(e^{-\theta x}\left(1+\frac{\theta x}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta-1} dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j+1}C_i^{\beta-1}C_j^{\alpha i+\alpha-1}C_k^{j+1} \times \frac{(-1)^i\theta^j}{(1+\theta)^{\alpha i+\alpha-1}} \times \Gamma((k+r+1),(\theta(\alpha i+\alpha-1)+\delta)t)$$

where $\Gamma(n,z)=(n-1)!e^{-z}\sum_{l=0}^{n-1}\frac{z^l}{l!}$

***proof* :**

$$\int_t^{\infty} x^r(1+x)e^{-\delta x}\left(e^{-\theta x}\left(1+\frac{\theta x}{(1+\theta)}\right)\right)^{\alpha-1}$$
$$\times\left[1-\left(e^{-\theta x}\left(1+\frac{\theta x}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta-1} dx$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j+1} C_i^{\beta-1}C_j^{\alpha i+\alpha-1}C_k^{j+1}\frac{(-1)^i\theta^j}{(1+\theta)^{\alpha i+\alpha-1}}(k+r)!$$
$$\times\exp[-(\theta(\alpha i+\alpha-1)+\delta)t]\times\sum_{l=0}^{k+r}\frac{(\theta(\alpha i+\alpha-1)+\delta)^l}{l!}$$

$\Rightarrow E(X^r|X>t)=\frac{\alpha\beta\theta^2}{(1+\theta)}L(\beta,\alpha,\theta,r,\delta,t)$

### *C. Quantile Function*

The quantile function $Q(p)$ of the proposed distribution is the solution of equation

$$F(Q(p))=p$$
$$\Rightarrow\left[1-\left\{e^{-\theta Q(p)}\left(1+\frac{\theta Q(p)}{1+\theta}\right)\right\}^{\alpha}\right]^{\beta}=p;\quad 0<p<1.$$
$$\Rightarrow e^{-\theta Q(p)}(1+\theta+\theta Q(p))=(1+\theta)(1-p^{1/\beta})^{1/\alpha}$$
$$\text{put}\quad Z(p)=-1-\theta-\theta Q(p)\quad for\quad 0<p<1$$
$$\text{then}\quad Z(p)e^{Z(p)}=-(1+\theta)e^{-(1+\theta)}(1-p^{1/\beta})^{1/\alpha}$$

So, the solution of this is

$$Z(p)=W\left[-(1+\theta)e^{-(1+\theta)}(1-p^{1/\beta})^{1/\alpha}\right];\ 0<p<1$$

where $W(\cdot)$ is Lambert $W$ function and also

$$Q(p)=-1-\frac{1}{\theta}-\frac{1}{\theta}W\left(-(1+\theta)e^{-(1+\theta)}(1-p^{1/\beta})^{1/\alpha}\right);\quad 0<p<1. \tag{8}$$

Now, on putting $p=1/2$, in equation (8), we get the median ($\eta_d$) of the proposed model.

### *D. Mean Deviation*

After considering the measure of central tendencies i.e. mean and median, we have also derived the measure of scatteredness in terms of mean deviation about the mean ($\mu_M$) and median ($\eta_d$). The mean deviation about mean ($\delta_1(x)$) and mean deviation about median ($\delta_2(x)$) and can be defined as

$$\delta_1(x)=\int_0^{\infty}|X-\mu_M|f(x)dx$$
$$\delta_2(x)=\int_0^{\infty}|X-\eta_d|f(x)dx$$

respectively. Then, for our proposed model

$$\delta_1(x)=\int_0^{\mu_M}(\mu_M-x)f(x)dx+\int_{\mu_M}^{\infty}(x-\mu_M)f(x)dx$$
$$=2\mu_M F(\mu_M)-2\mu_M+2\int_{\mu_M}^{\infty}xf(x)dx$$

and similarly,

$$\delta_2(x)=\int_0^{\eta_d}(\eta_d-x)f(x)dx+\int_{\eta_d}^{\infty}(x-\eta_d)f(x)dx$$
$$=\eta_d F(\eta_d)-\int_0^{\eta_d}xf(x)dx-\eta_d[1-F(\eta_d)]$$
$$+\int_{\eta_d}^{\infty}xf(x)dx$$
$$=-\mu_M+2\int_{\eta_d}^{\infty}xf(x)dx.$$

thus, by Lemma 3.2

$$\int_{\mu_M}^{\infty}xf(x)dx=\frac{\alpha\beta\theta^2}{(1+\theta)}L(\beta,\alpha,\theta,1,\theta,\mu_M)$$
$$\int_{\eta_d}^{\infty}xf(x)dx=\frac{\alpha\beta\theta^2}{(1+\theta)}L(\beta,\alpha,\theta,1,\theta,\eta_d)$$
$$\delta_1(x)=2\mu_M f(\mu_M)-2\mu_M+\frac{\alpha\beta\theta^2}{(1+\theta)}L(\beta,\alpha,\theta,1,\theta,\mu_M)$$
$$\delta_2(x)=-\mu_M+\frac{\alpha\beta\theta^2}{(1+\theta)}L(\beta,\alpha,\theta,1,\theta,\eta_d)$$

### *E. Generating Functions*

Let $X$ denotes a random variable follow the proposed distribution. Then from Lemma 3.1, the moment generating function is defined as

$$M_X(\xi)=E(e^{\xi X})=\frac{\alpha\beta\theta^2}{(1+\theta)}K(\beta,\alpha,\theta,0,\theta-\xi);\quad \xi<\theta$$

Similarly, the characteristic function of proposed model is given below

$$\phi_X(\xi)=E(e^{i\xi X})=\frac{\alpha\beta\theta^2}{(1+\theta)}K(\beta,\alpha,\theta,0,\theta-i\xi);\ \xi\in R$$

and the corresponding cumulant generating function is

$$K_X(\xi)=\ln\left(\frac{\alpha\beta\theta^2}{1+\theta}\right)+\ln K(\beta,\alpha,\theta,0,\theta-i\xi)\ \ \forall\ \xi<\theta$$

### *F. Order Statistics*

Let us define

$$V(x)=\left(e^{-\theta x}\left(1+\frac{\theta x}{1+\theta}\right)\right) \tag{9}$$

Then, the CDF and PDF of proposed model is

$$F(x)=[1-V^{\alpha}(x)]^{\beta}$$
$$f(x)=\frac{\alpha\beta\theta^2(1+x)e^{-\theta x}}{(1+\theta)}V^{\alpha-1}(x)[1-V^{\alpha}(x)]^{\beta-1}$$

respectively.
Now, the PDF $f_{k:n}(x)$ of the $k^{th}$ order statistic $X_k$ of a

random sample $x_1, x_2, ..., x_n$ of size n from the proposed distribution having PDF (5) is obtained as follows

$$\begin{aligned}
f_{k:n}(x) =& \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x) \\
=& \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}(-1)^l C_l^{n-k}[F(x)]^{k+l-1}f(x) \\
=& \frac{n!}{(k-1)!(n-k)!}\frac{\alpha\beta\theta^2(1+x)e^{-\theta x}}{(1+\theta)}\sum_{l=0}^{n-k}(-1)^l C_l^{n-k} \\
& \times V^{\alpha-1}(x)[1-V^{\alpha}(x)]^{\beta(k+l)-1} \\
=& \frac{n!}{(k-1)!(n-k)!}\frac{\alpha\beta\theta^2(1+x)e^{-\theta x}}{(1+\theta)}\sum_{l=0}^{n-k}\sum_{m=0}^{\infty}(-1)^{l+m} \\
& \times C_l^{n-k} C_m^{\beta(k+l)-1}[V^{\alpha(1+m)-1}(x)]
\end{aligned}$$

### G. Renyi Entropy

An entropy is a popular measure of uncertainty and proposed by various authors. The Renyi entropy (Renyi, A. (1961)) is one of the famous measure of uncertainty and defined as

$$J_R(\gamma) = \frac{1}{1-\gamma}\ln[\int f^{\gamma}(x)dx]; \quad \gamma > 0, \gamma \neq 1.$$

Now, the Renyi entropy for the random variable $X$ having proposed distribution with PDF (5) is calculated as follows

$$\begin{aligned}
\int f^{\gamma}(x)dx =& \left(\frac{\alpha\beta\theta^2}{(1+\theta)}\right)^{\gamma}\int_0^{\infty}(1+x)^{\gamma}e^{-\gamma\theta x}[V(x)]^{\gamma(\alpha-1)} \\
& \times[1-V^{\alpha}(x)]^{(\beta-1)\gamma}dx \\
=& \left(\frac{\alpha\beta\theta^2}{(1+\theta)}\right)^{\gamma}\sum_{i=0}^{\infty}C_i^{\gamma(\beta-1)}(-1)^i\int_0^{\infty}[V(x)]^{i+\gamma\alpha-\gamma} \\
& \times(1+x)^{\gamma}e^{-\gamma\theta x}dx \\
=& \left(\frac{\alpha\beta\theta^2}{(1+\theta)}\right)^{\gamma}\sum_{i=0}^{\infty}C_i^{\gamma(\beta-1)}(-1)^i \\
& \times\int_0^{\infty}\left(1+\frac{\theta x}{1+\theta}\right)^{\gamma\alpha-\gamma+i}(1+x)^{\gamma}e^{(-\gamma\theta x+\gamma\alpha-\gamma+i)}dx
\end{aligned}$$

$$\begin{aligned}
\int f^{\gamma}(x)dx =& \left(\frac{\alpha\beta\theta^2}{(1+\theta)}\right)^{\gamma}\sum_{i=0}^{\infty}C_i^{\gamma(\beta-1)}(-1)^{i+j} \\
& \times\sum_{j=0}^{\infty}C_j^{\gamma(\alpha-1)+i}\int_0^{\infty}\left(\frac{\theta x}{1+\theta}\right)^j \\
& \times\sum_{k=0}^{\infty}(-1)^k C_k^{\gamma}x^k e^{-\gamma(\theta x-\alpha+1)+i}dx
\end{aligned}$$

$$\begin{aligned}
\text{or } \int f^{\gamma}(x)dx =& \left(\frac{\alpha\beta\theta^2}{(1+\theta)}\right)^{\gamma}\left(\frac{\theta}{1+\theta}\right)^j \\
& \times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}C_i^{\gamma(\beta-1)} \\
& \times C_j^{\gamma(\alpha-1)+i}C_k^{\gamma}\exp[-(x(\alpha-1)-i)] \\
& \times\frac{\Gamma(k+j+1)}{(\gamma\theta)^{k+j+1}}
\end{aligned}$$

Thus the final expression of Renyi entropy is

$$J_R(\gamma) = \frac{1}{1-\gamma}\left[\gamma\ln\left(\frac{\alpha\beta\theta^2}{1+\theta}\right)+j\ln\left(\frac{\theta}{1+\theta}\right)+\ln\xi_c\right]$$

where $\xi_c = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}C_i^{\gamma(\beta-1)}C_j^{\gamma(\alpha-1)+i} \times C_k^{\gamma}\exp[-(x(\alpha-1)-i)]\frac{\Gamma(k+j+1)}{(\gamma\theta)^{k+j+1}}$

## IV. Estimation of Parameters

After the selection of an appropriate model, the next task is to obtain the estimate of the unknown parameters of the model. There are a number of methods discussed in statistical literature out of which we are considering the maximum likelihood method of estimation for the parameters $\beta,\ \alpha$ and $\theta$ of the proposed distribution.

**Maximum Likelihood Method of Estimation:**
The maximum likelihood method of estimation is a cleaver method of estimation as we try to obtain values of the parameters for which the sample in hand has the highest probability to come in hand. It has several interesting desirable properties such as it provides consistent estimators, sufficient statistics (if it is/ are exists), etc. Let us consider a random sample of size $n$ from the proposed distribution, then its likelihood function is given by

$$\begin{aligned}
L =& \prod_{i=1}^{n}f(x_i;\theta) \\
=& \left(\frac{\alpha\beta\theta^2}{1+\theta}\right)^n\prod_{i=1}^{n}(1+x_i)^n e^{-\theta x_i}\prod_{i=1}^{n}\left(e^{-\theta x_i}\left(1+\frac{\theta x_i}{(1+\theta)}\right)\right)^{\alpha-1} \\
& \times\prod_{i=1}^{n}\left[1-\left(e^{-\theta x_i}\left(1+\frac{\theta x_i}{(1+\theta)}\right)\right)^{\alpha}\right]^{\beta-1}.
\end{aligned}$$

And hence, the log-likelihood function is obtained as follows

$$\begin{aligned}
\ln L =& n\ln\left(\frac{\alpha\beta\theta^2}{1+\theta}\right)+\sum_{i=1}^{n}\ln(1+x_i)-\theta\sum_{i=1}^{n}x_i \\
& +(\alpha-1)\sum_{i=1}^{n}\ln(V(x_i))+(\beta-1)\sum_{i=1}^{n}\ln[1-V^{\alpha}(x_i)]
\end{aligned} \tag{10}$$

where, $V(x_i)$ is defined in (9).
Now, differentiating (10) with respect to $\theta,\ \alpha$ & $\beta$ and equated

to zero, we get

$$\begin{aligned}
\frac{\partial \ln L}{\partial \theta} &= 0 \\
&\Rightarrow \frac{n(\theta+2)}{\theta(1+\theta)} - \sum_{i=1}^{n} x_i + \frac{(\alpha-1)}{(1+\theta)^2} \sum_{i=1}^{n} e^{-\theta x_i} x_i \\
&\quad \times [1-(1+\theta)(\theta x_i+\theta+1)] \\
&\quad + \frac{(\beta-1)}{(1+\theta)^2} \sum_{i=1}^{n} \frac{\alpha V^{\alpha-1}(x_i)}{V^{\alpha}(x_i)-1} e^{-\theta x_i} x_i \\
&\quad \times [1-(1+\theta)(\theta x_i+\theta+1)] = 0 \\
\frac{\partial \ln L}{\partial \alpha} &= 0 \\
&\Rightarrow \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(V(x_i)) \\
&\quad + \alpha(\beta-1) \sum_{i=1}^{n} \frac{V^{\alpha}(x_i)\ln(V(x_i))}{1-V^{\alpha}(x_i)} = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ln L}{\partial \beta} &= 0 \\
&\Rightarrow \frac{n}{\beta} + \sum_{1}^{n} \ln(1-V(x_i)) = 0.
\end{aligned}$$

The simultaneous solution of the above likelihood equations constitutes MLEs of $\theta,\ \alpha$ & $\beta$. However, analytical solutions are not possible. Therefore, we use an approximation technique to solve the above normal equations with the help of R Core Team (2020) software.

## V. Real Data Application

Here, we have considered two real datasets to show the applicability of the proposed distribution with two different models belongs to the same family of distributions which are Lindley (Lindley, D. V. (1958)) and new generalized Lindley distribution (NGLD) (Elbatal, I., Merovci, F., & Elgarhy, M. (2013)) with PDF

$$f(x) = \frac{e^{-\theta x}}{1+\theta}\left(\frac{\theta^{\alpha+1}x^{\alpha-1}}{\Gamma\alpha} + \frac{\theta^{\beta}x^{\beta-1}}{\Gamma\beta}\right); \quad x>0$$
$$\theta,\ \alpha,\ \beta > 0.$$

We have considered two real datasets. The first dataset shows a sample of remission times (measured in months) of 128 bladder cancer patients and proposed by Lee, E. T. & Wang, J. (2003). The second dataset is the survival times (measured in days) of 72 guinea pigs infected with virulent tubercle bacilli and reported by Bjerkedal, T. (1960). We have considered these datasets because the NGLD is a three-parameters model and was reported by Elbatal, I., Merovci, F., & Elgarhy, M. (2013) shows that it is a more suitable model for the considered datasets. So, we want to compare the proposed model with three-parameters NGLD.

The various criterion like p-value, AIC (Akaike Information Criterion) and BIC (Bayesian information criterion) are used to check the fitting of the distributions. Also, we have calculated the negative of log-likelihood value $(-\ln L)$and KS (Kolmogorov-Smirnov) test statistic.

Firstly, we consider the p-values for checking which models are fitted to the considered dataset and after that we calculate the other mentioned criterion to know which model is more suitable for the data set among the considered model. It is worthless to mentioned here that the smaller values of AIC, BIC, KS test statistic and $-\ln L$ indicate a better fit of distributions. We have also calculated the MLEs of parameters for various distributions. All these values are given in Table I.

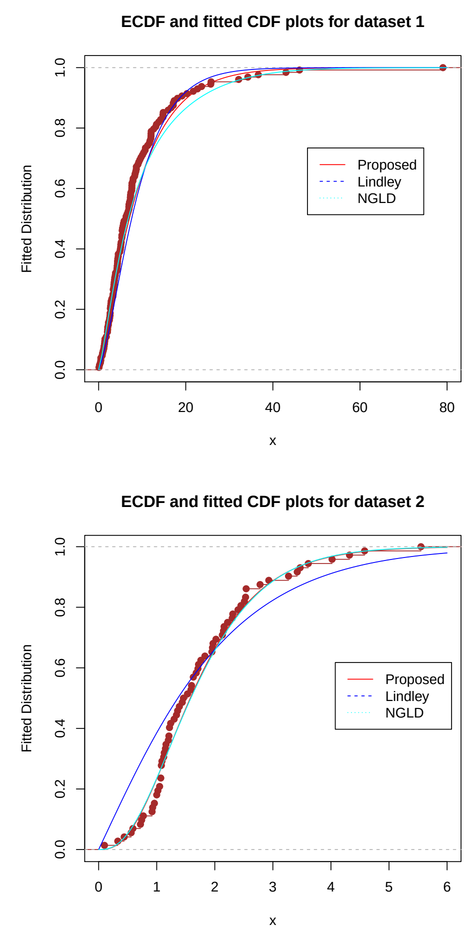


Fig. 3. ECDF plots of considered datasets.

| Data set | Distribution | ML Estimates $\theta$ | $\alpha$ | $\beta$ | KS Statistic | p-value | AIC | BIC | $\ln L$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | Lindley | 0.196 | - | - | 0.074 | 0.083 | 841.06 | 843.892 | 839.04 |
| | NGLD | 0.18 | 4.679 | 1.324 | 0.081 | 0.391 | 831.501 | 840.057 | 825.501 |
| | Proposed | 0.365 | 0.388 | 0.852 | 0.067 | 0.283 | 830.194 | 838.750 | 824.194 |
| 2 | Lindley | 0.868 | - | - | 0.232 | 0.000 | 215.857 | 218.133 | 213.857 |
| | NGLD | 1.861 | 3.585 | 2.737 | 0.089 | 0.612 | 194.364 | 201.194 | 188.364 |
| | Proposed | 1.022 | 1.486 | 2.827 | 0.087 | 0.649 | 193.921 | 200.751 | 187.921 |

TABLE I

Table for ML estimate, Log likelihood value, KS statistic, p-value, AIC and BIC

From Table I, we can say that for dataset 1, all model fitted well at 5% level of significance and KS distance along with model selection criterion having least values of AIC and BIC

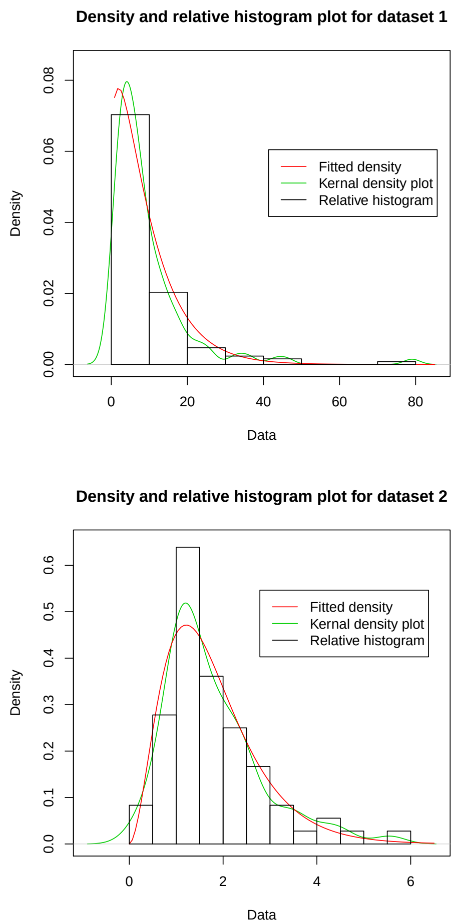


Fig. 4. Fitted density, Relative histogram and Kernel density plots for proposed distribution.

for the proposed distribution. Also, the negative of logarithmic of likelihood value is least for the proposed model.
Also, for dataset 2, only NGLD and proposed model fit well at desired level of significance (i.e. 5%), and similar result are obtained with same model selection criterion. Hence, we can say that proposed model fit well for both the datasets very well in comparison to other considered models. Graphics are widely used to gather information of any kind of data can be understand and interpreted by anyone easily. Fig. 3 and Fig. 4 empirical CDF and kernel density with histogram plot for the considered datasets respectively.

## VI. Conclusion

In this paper, we have generalized Lindley distribution with a single parameter using the generalization technique suggested by Cordeiro, G. M. & De Castro, M. (2011). The technique adds two additional shape parameters and hence the resulting distribution has three parameters. The new distribution, thus obtained is flexible in the sense of having different shapes of hazard rate function. It has IHR, DHR and bathtub shapes of hazard rate function. We have studied its various statistical properties like moments, conditional moments, reliability, quantile function, mean deviation about its mean and median, moment generating function, characteristic function, cumulant generating function and its order statistics. The maximum likelihood estimators of the parameters of this new distribution is obtained on the basis of the complete sample from it. Two suitable real datasets have been considered to show the applicability of the proposed distribution in real lifetime scenario. The criterion is taken as AIC, BIC, KS test statistic and log-likelihood along with empirical cumulative distribution function and kernel density with relative histogram plot in Fig. 3 and Fig. 4 respectively. The results shows that the same tools are also defined for Lindley distribution and NGLD. We observed from comparative Table I that our proposed distribution outperforms the other two distributions for the considered datasets in terms of AIC, BIC, KS test statistic and log-likelihood fitting criterion. Thus we recommend its further use to analyze different types of real datasets with a hope to get a better model.

## References


Aryal, G. R. & Yousof, H. M. (2017). The exponentiated generalized-G Poisson family of distributions. Stochastics and Quality Control, 32(1): 7-23.

Asgharzadeh, A., Bakouch, H. S., Nadarajah, S., & Sharafi, F. (2016). A new weighted Lindley distribution with application. Brazilian Journal of Probability and Statistics, 30(1): 1-27.

Bjerkedal, T. (1960). Acquisition of resistance in guinea pies infected with different doses of virulent tubercle bacilli. American Journal of Hygiene, 72(1): 130-48.

Cordeiro, G. M. & De Castro, M. (2011). A new family of generalized distributions. Journal of statistical computation and simulation, 81(7): 883-898.

Cordeiro, G. M., Ortega, E. E. M., & Daniel, C. C. D. C. (2013). The exponentiated generalized class of distributions. Journal of Data Science, 11: 1-27.

Dey, S., Nassar, M., & Kumar, D. (2017). Alpha logarithmic transformed family of distributions with application. Annals of Data Science, 4(4): 457-482.

Elbatal, I., Merovci, F., & Elgarhy, M. (2013). A new generalized Lindley distribution. Mathematical theory and Modeling, 3(13): 30-47.

Ghitany, M. E. & Al-Mutairi, D. K. (2008). Size-biased Poisson-Lindley distribution and its application. Metron-International Journal of Statistics, 66(3): 299-311.

Ghitany, M. E., Al-Mutairi, D. K., Balakrishnan, N., & Al-Enezi, L. J. (2013). Power Lindley distribution and associated inference. Computational Statistics & Data Analysis, 64(2013): 20-33.

Ghitany, M. E., Atieh, B., & Nadarajah, S. (2008). Lindley distribution and its application. Mathematics and computers in simulation, 78(4): 493-506.

Gomes, A. E., Da-Silva, C. Q., & Cordeiro, G. M. (2015). The exponentiated G Poisson model. Communications in Statistics-Theory and Methods, 44(20): 4217-4240.

Goyal, T., Rai, P. K., & Maurya, S. K. (2019). Classical and Bayesian studies for a new lifetime model in presence of type-II censoring. Communications for Statistical Applications and Methods, 26(4): 385-410.

Gupta, R. C., Gupta, P. L., & Gupta, R. D. (1998). Modeling failure time data by Lehmann alternatives. Communications in Statistics-Theory and Methods, 27: 887-904.

Kumar, D., Singh, U., & Singh, S. K. (2015). A method of proposing new distribution and its application to bladder cancer patients data. Journal of Statistics Applications and Probability Letters, 2(3): 235-245.

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. Journal of Hydrology, 46(1-2): 79-88.

Lee, E. T. & Wang, J. (2003). Statistical methods for survival data analysis, volume 476. John Wiley & Sons.

Lehmann, E. L. (1953). The power of rank tests. Annals of Mathematical statistics, 24:23-43.

Lindley, D. V. (1958). Fiducial distributions and Bayes' theorem. Journal of the Royal Statistical Society. Series B (Methodological), 102-107.

Mahdavi, A. & Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. Communications in Statistics-Theory and Methods, 46(13): 6543-6557.

Maurya, S., Kaushik, A., Singh, S., & Singh, U. (2017a). A new class of distribution having decreasing, increasing, and bathtub-shaped failure rate. Communications in Statistics-Theory and Methods, 46(20): 10359-10372.

Maurya, S. K., Kaushik, A., Singh, R. K., Singh, S. K., & Singh, U. (2016). A new method of proposing distribution and its application to real data. Imperial Journal of Interdisciplinary Research, 2(6): 1331-1338.

Maurya, S. K., Kaushik, A., Singh, S. K., & Singh, U. (2017b). A new class of exponential transformed Lindley distribution and its application to yarn data. International Journal of Statistics and Economics, 18(2): 135-151.

Maurya, S. K., Kumar, D., Singh, S. K., & Singh, U. (2018). One parameter decreasing failure rate distribution. International Journal of Statistics and Economics, 19(1): 120-138.

Maurya, S. K., Singh, S. K., & Singh, U. (2020). A new distribution with monotone and non-monotone shaped failure rate. Journal of Applied Probability and Statistics, 15(1): 1-21.

Mazucheli, J. & Achcar, J. A. (2011). The Lindley distribution applied to competing risks lifetime data. Computer methods and programs in biomedicine, 104(2): 188-192.

Nadarajah, S., Bakouch, H. S., & Tahmasbi, R. (2011). A generalized Lindley distribution. Sankhya B, 73(2): 331-359.

Nadarajah, S. & Kotz, S. (2006). The beta exponential distribution. Reliability engineering & system safety, 91(6): 689-697.

Pararai, M., Warahena-Liyanage, G., & Oluyede, B. O. (2015). An extended Lindley Poisson distribution. Journal of Mathematics and Statistical Science, 1(5): 167-198.

R Core Team (2020). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria.

Renyi, A. (1961). On measures of entropy and information. In Proceedings of the 4th Berkeley symposium on mathematical statistics and probability. Berkeley :University of California Press, 1: 547-561.

Sharma, V. K., Singh, S. K., Singh, U., & Agiwal, V. (2015). Inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data. Journal of Industrial and Production Engineering, 32(3): 162-173.

Shaw, W. & Buckley, I. (2007). The Alchemy of probability distribution: Beyond gram-charlier cornish-fisher expansions, and skew-normal and kurtotic-normal distribution. Research report.

Yadav, A.S., Kumar,D., Kumar, P., Singh, S.K. & Singh, U. (2020). A New Transmuted Lifetime Distribution: Statistical Application and Properties to Survival Data. Journal of Scientific Research, 64(1), 127-137.

Zeghdoudi, H. & Nedjar, S. (2016). Gamma Lindley distribution and its application. Journal of Applied Probability and Statistics, 11(1): 129-138.