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# Characterization and Estimation of Generalized Inverse Power Lindley Distribution

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***Abstract*—One of the most important branch of statistics is survival and reliability analysis. There are various lifetime models available in literature that have applications in these fields. However the researchers always keep searching for more flexible models that are effective in more complex situations. With the same motivation, an effort has been made to introduce a new distribution named as Generalized Inverse Power Lindley distribution that is expected to turnout more constructive while dealing with complex real life data. Various statistical properties of the model have been derived. The parameter estimates are obatined using Maximum Likelihood Estimation (MLE) technique. Simulation study has been conducted to assess the performance of maximum likleihood estimators. Applicability of the proposed model to the real data has been investigated by comparing the model with some existing distributions.**

***Index Terms*—Exponentiation, Simulation, Quantile function, Order statistics, Stochastic ordering**

## I. INTRODUCTION

Lindley distribution (LD) introduced by Lindley (1958) has drawn a lot of attention from researchers because of its broad applications in modeling the data having monotone hazard rates. A Random Variable (RV) $Z$ is said to have LD if its probability density function (pdf) is given by

$$f(z,\beta)=\frac{\beta^2}{1+\beta}\,(1+z)\,e^{-\beta z}\ ;\ z>0,\beta>0 \qquad (1)$$

LD has been extended by various researchers including Ghitany et al. (2008), Nadarajah et al. (2011), Ghitany et al. (2013).

It has been observed that most of the real-life systems have non-monotone(bathtub (BT) and upside down bathtub (UBT)) hazard rates. For the analysis of lifetime data having BT shape hazard functions, several lifetime models have been introduced by many authors (see Mudholkar et al. (1993), Xie et al. (1996), Xie et al. (2002)). Interestingly, the inverse class of the probability models turn out very useful for modeling UBT shape hazard functions. Sharma et al. (2015) introduced Inverse Lindley distribution (ILD) with pdf given by:

$$f(z,\beta)=\frac{\beta^2}{1+\beta}\left(\frac{1+z}{z^3}\right)e^{\frac{-\beta}{z}}\ ;\ z>0,\beta>0 \qquad (2)$$

Sharma et al. (2016) extended ILD by adding a parameter and obtained a Generalized Inverse Lindley Distribution (GILD). Note that Barco et al. (2016) also generalized ILD by taking the transformation $Z=Y^{\frac{1}{\alpha}}$ where $Y$ follows ILD. The pdf of GILD is given by:

$$f(z,\beta)=\frac{\alpha\beta^2}{1+\beta}\left(\frac{1+z^\alpha}{z^{2\alpha+1}}\right)e^{\frac{-\beta}{z^\alpha}}\ ;\ z>0,\alpha>0,\beta>0 \quad (3)$$

In this article a new three parameter distribution named as Generalized Inverse Power Lindley Distribution (GIPLD) has been introduced. The proposed distribution is obtained by using the transformation $H(z)=[G(z)]^\theta$ where $G(z)$ is a CDF and $\theta$ is a positive real number. The new distribution thus obtained involves GILD and ILD as its sub-models for $\theta=1$ and $\alpha=\theta=1$ respectively. A RV $Z$ is said to follow GIPLD if its cumulative distribution function (CDF) is given by:

$$G(z)=\left[\left(1+\frac{\beta}{1+\beta}\frac{1}{z^\alpha}\right)e^{\frac{-\beta}{z^\alpha}}\right]^\theta;\quad z>0\ s(\alpha,\,\beta,\,\theta)>0 \qquad (4)$$

and the corresponding pdf as:

$$g(z)=\frac{\alpha\beta^2\theta}{1+\beta}\left(\frac{1+z^\alpha}{z^{2\alpha+1}}\right)e^{\frac{-\theta\beta}{z^\alpha}}\left[1+\frac{\beta}{1+\beta}\frac{1}{z^\alpha}\right]^{\theta-1};\qquad (5)$$
$$z>0\,,\,(\alpha,\,\beta,\,\theta)>0$$

where $(\alpha,\beta,\theta)$ are the parameters of the distribution, $\beta$ being scale while as $\alpha,\theta$ are the shape parameters.

The aim of this article is to obtain a more flexible model that exhibits both monotone and non-monotone behavior and thus motivates us to apply single model for two distinct behaviors

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of hazard rate. Adding a parameter $\theta$ to GILD adds more flexibility to the new distribution that competes well with other existing lifetime models. The rest of article is organized as follows: Reliability measures, moments, Renyi entropy, distribution of order statistics and stochastic ordering are presented in section II, III,IV,V,VIrespectively. Measures such as quantile function is given in section VII. The method of MLE has been discussed in section VIII. Section IX consists a simulation study to compare the performance of ML estimators followed by data analysis in section X. The article is concluded in section XI.

## II. Reliability measures

The survival function of $Z$ is obtained as:

$$S(z) = 1 - \left[\left(1 + \frac{\beta}{1+\beta}\frac{1}{z^{\alpha}}\right) e^{\frac{-\beta}{z^{\alpha}}}\right]^{\theta}, \tag{6}$$

and the corresponding hazard function as:

$$h(z) = \frac{\frac{\alpha\beta^2\theta}{1+\beta}\left(\frac{1+z^{\alpha}}{z^{2\alpha+1}}\right) e^{\frac{-\theta\beta}{z^{\alpha}}}\left[1 + \frac{\beta}{1+\beta}\frac{1}{z^{\alpha}}\right]^{\theta-1}}{1 - \left[\left(1 + \frac{\beta}{1+\beta}\frac{1}{z^{\alpha}}\right) e^{\frac{-\beta}{z^{\alpha}}}\right]^{\theta}} \quad , z > 0. \tag{7}$$

Further the Reverse Hazard Rate (RHR) of $Z$ is readily obtained as:

$$\lambda(z) = \frac{\frac{\alpha\beta^2\theta}{1+\beta}\left(\frac{1+z^{\alpha}}{z^{2\alpha+1}}\right) e^{\frac{-\theta\beta}{z^{\alpha}}}\left[1 + \frac{\beta}{1+\beta}\frac{1}{z^{\alpha}}\right]^{\theta-1}}{\left[\left(1 + \frac{\beta}{1+\beta}\frac{1}{z^{\alpha}}\right) e^{\frac{-\beta}{z^{\alpha}}}\right]^{\theta}} \quad , z > 0. \tag{8}$$

The plot of density function, survival function, hazard rate and RHR for different values of parameters are displayed in Figure 1-4.

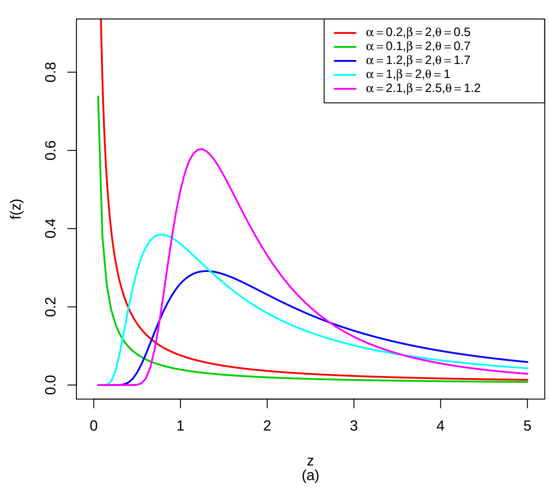


Fig. 1: Density plot of GIPLD($\alpha, \beta, \theta$)

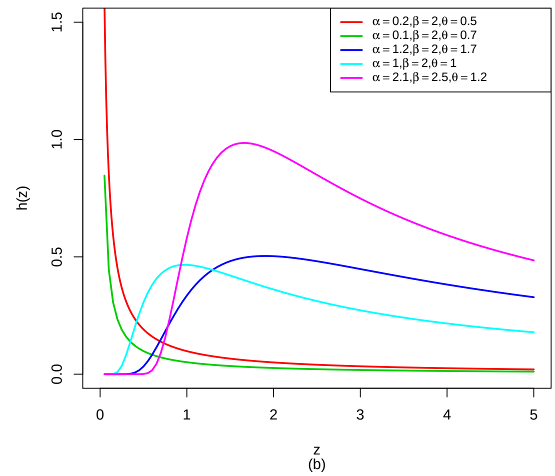


Fig. 2: Hazard rate for GIPLD($\alpha, \beta, \theta$)

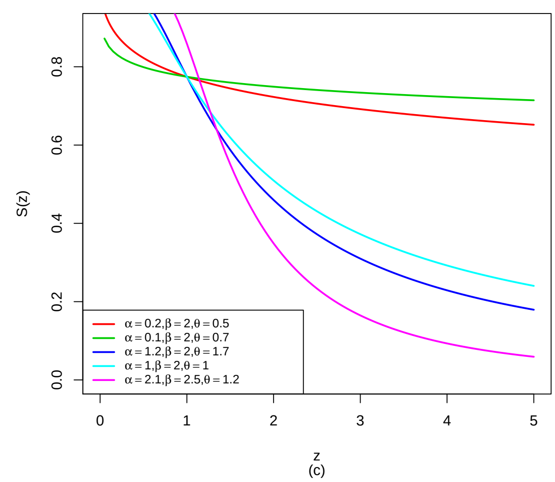


Fig. 3: Survival function of GIPLD($\alpha, \beta, \theta$)

## III. Moments

Moments play a very important role in probability distributions. Moments are the constants of a population and these constants help in deciding the characteristics of the population and on the basis of these characteristics a population is discussed.

*Theorem 1:* Let $Z$ be a random variable having pdf (5), then the $r^{th}$ raw moment of $Z$ is

$$\mu_r^{'} = E(Z^r) = (\beta\theta)^{\frac{r}{\alpha}} \sum_{i=0}^{\infty} \binom{\theta-1}{i} \frac{1}{(\theta(\beta+1))^{i+1}} \left(i + 1 - \frac{r}{\alpha} + \theta\beta\right) \Gamma\left(i + 1 - \frac{r}{\alpha}\right).$$

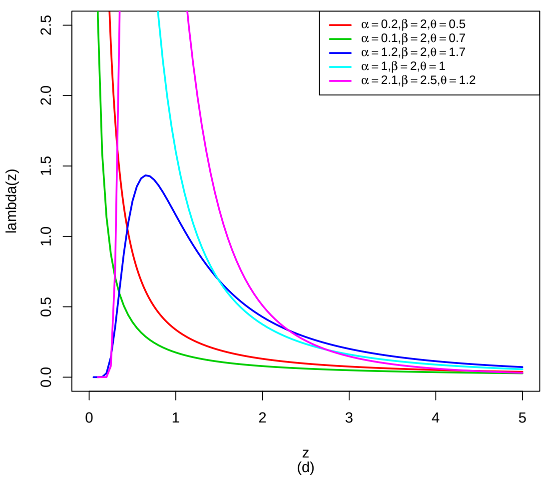


Fig. 4: Reverse hazard rate of GIPLD($\alpha, \beta, \theta$)

*Proof:* The $r^{th}$ raw moment of $Z$ is given by:

$$E(Z^r) = \int_0^\infty z^r g_\theta(z)dz$$
$$= \frac{\alpha\beta^2\theta}{1+\beta}\int_0^\infty z^r\left(\frac{1+z^a}{z^{2\alpha+1}}\right)e^{\frac{-\beta\theta}{z^\alpha}}\left[1+\frac{\beta}{1+\beta}\frac{1}{z^\alpha}\right]^{\theta-1}dz \quad (9)$$

Using the binomial expansion,

$$\left[1+\frac{\beta}{1+\beta}\frac{1}{z^\alpha}\right]^{\theta-1} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}\left(\frac{\beta}{1+\beta}\frac{1}{z^\alpha}\right)^i.$$

we get,

$$E(Z^r) = \frac{\alpha\beta^2\theta}{1+\beta}\sum_{i=0}^{\infty}\binom{\theta-1}{i}\left(\frac{\beta}{1+\beta}\right)^i\left[\int_0^\infty \frac{1}{z^{\alpha(2+i)-r+1}} e^{\frac{-\beta\theta}{z^\alpha}}dz + \int_0^\infty \frac{1}{z^{\alpha(i+1)-r+1}}e^{\frac{-\beta\theta}{z^\alpha}}dz\right] \quad (10)$$

Let $t = z^\alpha$ and using the definition of inverse gamma distribution (10) reduces to

$$E(Z^r) = \frac{\beta^2\theta}{1+\beta}\sum_{i=0}^{\infty}\binom{\theta-1}{i}\left(\frac{\beta}{1+\beta}\right)^i\left[\frac{\Gamma(2+i-\frac{r}{\alpha})}{(\theta\beta)^{2+i-\frac{r}{\alpha}}} + \frac{\Gamma(i+1-\frac{r}{\alpha})}{(\theta\beta)^{i+1-\frac{r}{\alpha}}}\right]$$

Therefore,

$$\mu_r^{'} = (\beta\theta)^{\frac{r}{\alpha}}\sum_{i=0}^{\infty}\binom{\theta-1}{i}\frac{1}{(\theta(\beta+1))^{i+1}}\left(i+1-\frac{r}{\alpha}+\theta\beta\right)\Gamma\left(i+1-\frac{r}{\alpha}\right) \quad (11)$$

For $r^{th}$ moment to exist, the constraint $\alpha > r$ must be satisfied. Note that for $\theta = 1$, (11) reduces to $r^{th}$ moment of GILD. ■

## IV. Renyi entropy

Entropies quantify the diversity, uncertainty, or randomness of a system. For a given probability distribution, Renyi entropy is given by:

$$e(\gamma) = \frac{1}{1-\gamma}log\left[\int g^\gamma(z)dz\right]$$

where $\gamma > 0$ *and* $\gamma \neq 1$

$$e(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^\infty \frac{\alpha\beta^2\theta}{1+\beta}\left(\frac{1+z^\alpha}{z^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{z^\alpha}}\left(1+\frac{\beta}{1+\beta}\frac{1}{z^\alpha}\right)^{\theta-1}dz\right]$$

$$e(\gamma) = \frac{1}{1-\gamma}log\left(\frac{\alpha\beta^2\gamma}{1+\beta}\right)^\gamma\sum_{i=0}^{\infty}\binom{\gamma\theta-\gamma}{i}\left(\frac{\beta}{1+\beta}\right)^i\sum_{j=0}^{\infty}\binom{\gamma}{j}\int_0^\infty e^{-\frac{\gamma\beta\alpha}{z^\alpha}}\frac{1}{z^{\alpha(i+j+\gamma)+\gamma}}dz \quad (12)$$

After simplification, (12) becomes,

$$e(\gamma) = \frac{1}{1-\gamma}log\left(\frac{\alpha^{\gamma-1}\beta^{2\gamma}\theta^\gamma}{(1+\beta)^\gamma}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\gamma\theta-\gamma}{i}\binom{\gamma}{j}\left(\frac{\beta}{1+\beta}\right)^i\frac{\Gamma\left(i+j+\gamma+\frac{\gamma-1}{\alpha}\right)}{(\theta\beta\gamma)^{i+j+\gamma+\frac{\gamma-1}{\alpha}}}$$

## V. Distribution of Order Statistic

The pdf of $k^{th}$ order statistic of $Z$ is

$$g_k(z) = \frac{n!g(z)}{(k-1)!(n-k)!}(G(z))^{k-1}(1-G(z))^{n-k}. \quad (13)$$

where $g(z)$ and $G(z)$ denotes the pdf and cdf respectively.

Using $(1-z)^n = \sum_{i=0}^{\infty}\binom{n}{i}(-1)^i(z)^i$ in (13), we get

$$g_k(z) = \frac{n!g(z)}{(k-1)!(n-k)!}\sum_{i=0}^{\infty}\binom{n-k}{i}(-1)^i(G(z))^{i+k-1}g(z).$$

$$g_k(z) = \frac{n!}{(k-1)!(n-k)!}\sum_{i=0}^{\infty}(-1)^i\binom{n-k}{i}\frac{\alpha\beta^2\theta}{1+\beta}\left(\frac{1+z^\alpha}{z^{2\alpha+1}}\right)e^{-\frac{\theta\beta}{z^\alpha}}\left(1+\frac{\beta}{1+\beta}\frac{1}{z^\alpha}\right)^{\theta i+\theta k-1}.$$

Therefore,

$$g_k(z) = \frac{\alpha\theta\beta^{j+2}n!}{(1+\beta)^{j+1}(k-1)!(n-k)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^i\binom{n-k}{i}\binom{\theta i+\theta k-1}{j}\left(\frac{1+z^\alpha}{z^{(j+2)\alpha+1}}\right)e^{-\frac{\theta\beta}{z^\alpha}} \quad (14)$$

## VI. Stochastic Ordering

Stochastic ordering of a continuous random variable is an important tool for judging the comparative behavior. A random variable $Z$ is said to be greater than $Y$ in the:
(a) Stochastic order ($Y \leq_{st} Z$) if $G_Z(z) \leq G_Y(z) \forall z$
(b) Hazard rate order ($Y \leq_{hr} Z$) if $h_Z(z) \leq h_Y(z) \forall z$
(c) Mean residual order ($Y \leq_{mlr} Z$) if $m_Z(z) \leq m_Y(z) \forall z$
(d) Likelihood ratio order ($Y \leq_{lr} Z$) if $\frac{g_Z(z)}{g_Y(z)}$ is an increasing function of $z$.

The following results by Shaked and Shantikumar (1994) are well known:

$$\begin{array}{c} Z \leq_{lr} Y \Rightarrow Z \leq_{hr} Y \Rightarrow Z \leq_{mrl} Y \\ \Downarrow \\ Z \leq_{st} Y \end{array}$$

The following theorem shows that GIPLD is ordered with respect to "likelihood ratio" ordering.

*Theorem 2:* Let $Y \sim GIPLD(\alpha_1, \beta_1, \theta_1)$ and $Z \sim GIPLD(\alpha_2, \beta_2, \theta_2)$. .If $\beta_1 = \beta_2$ and $\theta_2 \geq \theta_1$ (or if $\beta_2 \geq \beta_1$ and $\theta_1 = \theta_2$ ) then $(Y \leq_{lr} Z) \forall z$.

*Proof:*
We have

$$\frac{g_Z(z)}{g_Y(z)} = \frac{\alpha_2\beta_2^2\theta_2(1+\beta_1)}{\alpha_1\beta_1^2\theta_1(1+\beta_2)}\left(\frac{1+z^{\alpha_2}}{1+z^{\alpha_1}}\right)\left(\frac{z^{2\alpha_1+1}}{z^{\alpha_2+1}}\right)$$
$$exp^{-\left(\frac{\theta_2\beta_2}{z^{\alpha_2}}-\frac{\theta_1\beta_1}{z^{\alpha_1}}\right)}.\left[\frac{\left(1+\frac{\beta_2}{1+\beta_2}\frac{1}{z^{\alpha_2}}\right)^{\theta_2-1}}{\left(1+\frac{\beta_1}{1+\beta_1}\frac{1}{z^{\alpha_1}}\right)^{\theta_1-1}}\right].$$

$$\frac{\partial}{\partial z} ln\frac{g_Z(z)}{g_Y(z)} = \frac{1}{1+z^{\alpha^2}} - \frac{1}{1+z^{\alpha_1}} + \frac{1}{z^{2\alpha_2+1}} - \frac{1}{z^{2\alpha_1+1}} + \frac{\theta_2\beta_2\alpha_2}{z^{2\alpha_2+1}} - \frac{\theta_1\beta_1\alpha_1}{z^{2\alpha_1+1}} - \frac{\alpha_2\beta_2(\theta_2-1)}{(1+\beta_2)z^{\alpha_2+1}}\frac{1}{\left(1+\frac{\beta_2}{1+\beta_2 z^{\alpha_2}}\right)} + \frac{\alpha_1\beta_1(\theta_1-1)}{(1+\beta_1)z^{\alpha_1+1}}\frac{1}{\left(1+\frac{\beta_1}{1+\beta_1 z^{\alpha_1}}\right)}. \quad (15)$$

Setting $\alpha_1 = \alpha_2$

**Case 1**: for $\beta_1 = \beta_2 = \beta$ and $\theta_2 \geq \theta_1$, $\frac{d}{dz}\left(ln\frac{g_Z(z)}{g_Y(z)}\right)$ is obtained as an increasing function of $z$.
**Case 2**: $\theta_1 = \theta_2 = \theta$ and $\beta_2 \geq \beta_1$, $\frac{d}{dz}\left(ln\frac{g_Z(z)}{g_Y(z)}\right)$ is obtained as an increasing function of $z$.

This implies that $Y \leq_{lr} Z \ \forall \ z$.Hence $Y \leq_{hr} Z$ , $Y \leq_{mrl} Z$ and $Y \leq_{st} Z$. ■

## VII. Quantile Function

*Theorem 3:* Let $Z$ follows (5), then the quantile function of $Z$ is

$$Q(u) = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left(-u^{\frac{1}{\theta}}(1+\beta)e^{-(1+\beta)}\right)\right]^{-\frac{1}{\alpha}}.$$

where $u \in (0,1)$ and $W_{-1}$ denote the negative branch of Lambert $W$ function.

*Proof:* The quantile function denoted by $Q(u)$ is the root of equation

$$\left[\left(1+\frac{\beta}{1+\beta}\frac{1}{Q(u)^\alpha}\right)e^{\frac{-\beta}{Q(u)^\alpha}}\right]^\theta = u \ \ ;0<u<1 \quad (16)$$

Multiplying (16) by $e^{-1-\beta}$ we get,

$$-\left[1+\beta+\frac{\beta}{Q(u)^\alpha}\right]e^{-\left(1+\beta+\frac{\beta}{Q(u)^\alpha}\right)} = -(1+\beta)u^{\frac{1}{\theta}}e^{-(1+\beta)}.$$

Using the Lambert $W$ function which is the solution of the equation $W(z)e^{W(z)} = z$, where $z$ is a complex number, we have

$$W\left(-u^{\frac{1}{\theta}}e^{-(1+\beta)}(1+\beta)\right) = -\left(1+\beta+\frac{\beta}{Q(u)^\alpha}\right)$$

The negative Lambert $W$ function of the real argument $-u(1+\beta)e^{1+\beta}$ is

$$W_{-1}\left(-u^{\frac{1}{\theta}}e^{-(1+\beta)}(1+\beta)\right) = -\left(1+\beta+\frac{\beta}{Q(u)^\alpha}\right).$$

which upon solving for $Q(u)$ results in

$$Q(u) = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left(-u^{\frac{1}{\theta}}(1+\beta)e^{-(1+\beta)}\right)\right]^{-\frac{1}{\alpha}}. \quad (17)$$

■

which is the quantile function of GIPLD. Median of GIPLD is evaluated using (17) as:

$$Q\left(\frac{1}{2}\right) = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left(-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}(1+\beta)e^{-(1+\beta)}\right)\right]^{-\frac{1}{\alpha}}$$

## VIII. Estimation

Let $z_1, .....z_n$ be a random sample of size $n$ from (5). The log-likelihood function $L(z, \Theta)$ for a vector of parameters $\Theta = (\alpha, \beta, \theta)^T$ is given by

$$L(z,\Theta) = n[\log\alpha + 2\log\beta + \log\theta - \log(1+\beta)] + \sum_{i=1}^{n}\log(1+z_i^\alpha) - (2\alpha+1)\sum_{i=1}^{n}log(z_i) - \theta\beta \sum_{i=1}^{n} z_i^{-\alpha}(\theta-1)\sum_{i=1}^{n}log\left[1+\frac{\beta}{1+\beta}\frac{1}{z_i^\alpha}\right].$$

The normal equations to estimate $\Theta = (\alpha, \beta, \theta)$ are :

$$\frac{\partial}{\partial\alpha}L(z,\Theta) = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{z_i^\alpha log z_i}{1+z_i^\alpha} - 2\sum_{i=1}^{n}log(z_i) + \theta\beta\sum_{i=1}^{n}(z_i^{-\alpha}) \log z_i(\theta-1)\sum_{i=1}^{n}\left[\frac{\frac{\beta}{1+\beta}(z_i^{-\alpha})log(z_i)}{1+\frac{\beta}{1+\beta}\frac{1}{z_i}^\alpha}\right] = 0$$

$$\frac{\partial}{\partial\beta}L(z,\Theta) = \frac{n(2+\beta)}{\beta(1+\beta)} - \theta\sum_{i=1}^{n}z_i^{-\alpha} + (\theta-1) \sum_{i=1}^{n}\left[\frac{1}{1+\frac{\beta}{1+\beta}\frac{1}{z_i^\alpha}}\right]\left[\frac{1}{z_i^\alpha}\frac{1}{(1+\beta^2)}\right] = 0$$

$$\frac{\partial}{\partial\theta}L(z,\Theta)=\frac{n}{\theta}-\beta\sum_{i=1}^{n}z_i^{-\alpha}+\sum_{i=1}^{n}log\left[1+\frac{\beta}{1+\beta}\frac{1}{z_i^{\alpha}}\right]=0$$

The above non-linear system of equations cannot be solved explicitly hence we use numerical iteration technique in order to find the estimate of $\Theta$. Since the maximum likelihood estimates for $\Theta$ are not in closed form we use the large sample behavior of maximum likelihood estimators to obtain the confidence intervals for model parameters. The asymptotic sampling distribution of $\hat{\Theta}$ is $N[\Theta,\Delta^{-1}]$ where $\Delta$ is the observed Fisher information matrix given by:

$$\Delta=\begin{bmatrix}\frac{\partial^2L}{\partial\alpha^2} & \frac{\partial^2L}{\partial\alpha\partial\beta} & \frac{\partial^2L}{\partial\alpha\partial\theta}\\ \frac{\partial^2L}{\partial\beta\partial\alpha} & \frac{\partial^2L}{\partial\beta^2} & \frac{\partial^2L}{\partial\beta\partial\theta}\\ \frac{\partial^2L}{\partial\theta\partial\alpha} & \frac{\partial^2L}{\partial\theta\partial\beta} & \frac{\partial^2L}{\partial\theta^2}\end{bmatrix} \tag{18}$$

or

$$\Delta=\begin{bmatrix}\widehat{Var(\alpha)} & \widehat{Cov(\alpha,\beta)} & \widehat{Cov(\alpha.\theta)}\\ \widehat{Cov(\beta,\alpha)} & \widehat{Var(\beta)} & \widehat{Cov(\beta,\theta)}\\ \widehat{Cov(\theta,\alpha)} & \widehat{Cov(\theta,\beta)} & \widehat{Var(\theta)}\end{bmatrix} \tag{19}$$

The second order derivatives for the parameters of GIPLD exist and are derived as:

$$\frac{\partial^2L}{\partial\alpha^2}=-\frac{n}{\alpha^2}+\sum_{i=1}^{n}\frac{z_i^{\alpha}(logz_i)^2}{(1+z_i^{\alpha})^2}-\theta\beta\sum_{i=1}^{n}z_i^{-\alpha}logz_i$$
$$+(\theta-1)\left(\frac{\beta}{1+\beta}\right)\sum_{i=1}^{n}\frac{z_i^{-\alpha}(logz_i)^2}{\left(1+\frac{\beta}{1+\beta}\frac{1}{z_i^{\alpha}}\right)^2}$$

$$\frac{\partial^2L}{\partial\beta^2}=\frac{-2n}{\beta^2}+\frac{n}{(1+\beta)^2}-\frac{(\theta-1)}{(1+\beta)^4}$$
$$\sum_{i=1}^{n}\left[\left(\frac{1}{1+\frac{\beta}{1+\beta}\frac{1}{z_i^{\alpha}}}\right)\frac{1}{z_i^{\alpha}}\right]^2(3+2\beta(1+z_i))$$

$$\frac{\partial^2L}{\partial\theta^2}=\frac{-n}{\theta^2}.$$
$$\frac{\partial^2L}{\partial\alpha\partial\theta}=\beta\sum_{i=1}^{n}(z_i^{-\alpha})logz_i-\frac{\beta}{1+\beta}\sum_{i=1}^{n}\left[\frac{z_i^{-\alpha}logz_i}{1+\frac{\beta}{1+\beta}\frac{1}{z_i^{\alpha}}}\right].$$
$$\frac{\partial^2L}{\partial\beta\partial\theta}=-\frac{\beta}{(1+\beta)^3}\sum_{i=1}^{n}\left(\frac{1}{z_i^{\alpha}}\right)^2.$$

$$\frac{\partial^2L}{\partial\beta\partial\alpha}=\theta\sum_{i=1}^{n}z_i^{-\alpha}logz_i-\frac{(\theta-1)}{(1+\beta)^2}\sum_{i=1}^{n}\frac{z_i^{-\alpha}logz_i}{\left[1+\frac{\beta}{1+\beta}\frac{1}{z_i^{\alpha}}\right]^2}.$$

The solutions of the above equations yield the asymptotic variance covariance of ML estimators for $\Theta$. The asymptotic confidence intervals for $\alpha,\beta,\theta$ is: $\hat{\alpha}\pm Z_{\frac{\alpha}{2}}\sqrt{V\hat{ar}(\alpha)}$ , $\hat{\beta}\pm Z_{\frac{\alpha}{2}}\sqrt{V\hat{ar}(\beta)}$ , $\hat{\theta}\pm Z_{\frac{\alpha}{2}}\sqrt{V\hat{ar}(\theta)}$

TABLE II: Maximum likelihood estimates of different models

| Model | Estimates | | |
|---|---|---|---|
| | $\alpha$ | $\beta$ | $\theta$ |
| GIPLD | 1.20 | 25.94 | 0.06 |
| MW | 0.96 | 0.001 | 0.27 |
| PLD | 0.79 | 0.58 | - |
| GLD | 0.74 | - | 0.36 |
| ILD | - | 2.05 | - |

## IX. Simulation Study

In this section, the performance of ML estimators for different sample sizes $(n = 25, 50, 100, 200, 300, 500)$ has been studied. The inverse CDF technique has been employed for data simulation using R software. The process was repeated 500 times for calculation of bias, variance and Mean Square Error (MSE). From Table I, it can be shown that for two parameter combinations of GIPLD, decreasing trend is being observed in average bias, variance and MSE as the sample size is increased. Hence, the performance of ML estimators is quite well, consistent in case of GIPLD.

## X. Data Analysis

In this section, real life data analysis is performed to illustrate the applicability of GIPLD. The data set represents the active repair times (hr) for an airborne communication transceiver. This data has been widely used by various authors and were initially used by Jorgensen (1982).

The proposed model has been compared with Modified weibull distribution (MW), Power Lindley distribution (PLD), Generalized Lindley distribution(GLD) and ILD using Akaike information criterion($AIC$) defined by $-2logL+2q$, corrected Akaike's information criterion($AICc$) defined by $AIC+\frac{2q(q+1)}{n-q-1}$, Bayesian information criterion($BIC$) defined by $-2logL+qlog(n)$. The proposed model is checked for goodness of fit test using Kolmogrov-Smirnov statistic defined by $KS = Max|F_0(X) - F_r(X)|$, where $F_0(X)$ is observed cummulative frequency and $F_r(X)$ is the theoretical frequency distribution. All the computations has been carried out using R software.The pdf*s* of compared distributions are :

$$f_{MW}(z)=(\beta+\theta\alpha z^{\alpha-1})e^{-\beta z-\theta z^{\alpha}}.$$
$$f_{PLD}(z)=\frac{\alpha\beta^2}{\beta+1}(1+z^{\alpha})z^{\alpha-1}e^{-\beta z^{\alpha}}.$$
$$f_{GLD}(z)=\frac{\theta\beta^2}{\beta+1}(1+z)\left[1-\left(1+\frac{\beta z}{1+\beta}e^{-\beta z}\right)\right]^{\theta-1}e^{-\beta z}.$$

The ML estimates along with standard error of parameters are displayed in Table II. Tble III displays $p$-value along with the above mentioned statistical values. It is quite apparent form Table III that the proposed model has least $-logL, AIC, AICC, BIC$ values and therefore competes well with other lifetime models.

TABLE I: Simulation study of ML estimators for GIPLD

| Parameter | n | $\alpha=0.1,\beta=0.9,\theta=0.3$ | | | $\alpha=0.4,\beta=1.5,\theta=0.1$ | | |
|---|---|---|---|---|---|---|---|
| | | **Bias** | **variance** | **MSE** | **Bias** | **Variance** | **MSE** |
| $\alpha$ | 25 | 0.712085 | 0.101666 | 0.608731 | 0.684085 | 0.341057 | 0.80903 |
| $\beta$ | | 1.605035 | 0.371255 | 2.947391 | 1.186668 | 0.351011 | 1.759191 |
| $\theta$ | | 0.091924 | 0.003196 | 0.011646 | 0.218817 | 0.001859 | 0.04974 |
| $\alpha$ | 50 | 0.584829 | 0.010782 | 0.352806 | 0.352148 | 0.016023 | 0.140031 |
| $\beta$ | | 1.256111 | 0.241674 | 1.819488 | 0.708452 | 0.573856 | 1.07576 |
| $\theta$ | | 0.094421 | 0.002579 | 0.011495 | 0.207594 | 0.003101 | 0.046196 |
| $\alpha$ | 100 | 0.682481 | 0.0068 | 0.472581 | 0.32123 | 0.022468 | 0.125657 |
| $\beta$ | | 0.870167 | 0.084289 | 0.84148 | 0.582663 | 0.216566 | 0.556063 |
| $\theta$ | | 0.077233 | 0.004805 | 0.01077 | 0.192033 | 0.005131 | 0.042007 |
| $\alpha$ | 200 | 0.581494 | 0.016392 | 0.354527 | 0.302592 | 0.007101 | 0.098663 |
| $\beta$ | | 0.83053 | 0.039215 | 0.728996 | 0.429972 | 0.128085 | 0.31296 |
| $\theta$ | | 0.060925 | 0.00208 | 0.005791 | 0.176234 | 0.00627 | 0.037328 |
| $\alpha$ | 300 | 0.58948 | 0.002064 | 0.349551 | 0.279823 | 0.004912 | 0.083213 |
| $\beta$ | | 0.761678 | 0.065107 | 0.645261 | 0.254516 | 0.056483 | 0.121262 |
| $\theta$ | | 0.036123 | 0.004829 | 0.006134 | 0.124068 | 0.003609 | 0.019002 |
| $\alpha$ | 500 | 0.456541 | 0.000728 | 0.209158 | 0.261378 | 0.002637 | 0.070955 |
| $\beta$ | | 0.644483 | 0.059861 | 0.475219 | 0.189692 | 0.053695 | 0.089679 |
| $\theta$ | | 0.018788 | 0.00425 | 0.004603 | 0.102064 | 0.002812 | 0.013229 |

TABLE III: Comparison of GIPLD and other models

| Model | -LogL | AIC | AICC | BIC | K-S statistic | *p*-value |
|---|---|---|---|---|---|---|
| GIPLD | 89.45 | 184.91 | 185.57 | 189.97 | 0.094 | 0.86 |
| MW | 95.51 | 197.02 | 197.69 | 202.09 | 0.19 | 0.52 |
| PLD | 95.94 | 195.88 | 196.21 | 199.26 | 0.13 | 0.46 |
| GLD | 97.91 | 199.82 | 200.15 | 203.2 | 0.16 | 0.22 |
| ILD | 90.05 | 195.11 | 196.21 | 193.8 | 0.09 | 0.73 |

## XI. Conclusion

In this study, a three parameter distribution named as GIPLD has been proposed. Some statistical properties such as reliability measures, moments, quantile, stochastic ordering, Renyi entropy of the proposed distribution has been discussed. The parameter estimation is approached by method of maximum likelihood estimation. Confidence intervals for the model parameters have also been derived. Monte carlo simulations were performed to investigate the performance of Maximum likelihood. The results of the simulation revealed the bias, MSE decreases as sample size is increased. Finally, application of the proposed distribution were illustrated using real data set.

## References


Barco, Mazucheli, & Janerio (2016). The inverse power lindley distribution. Communication in statistics-Simulation and Computation, 46(8), 6308-6323.

Ghitany, M., Atieh, B., & Nadarajah , S. (2008). Lindley distribution and its Application. Mathematics Computing and Simulation, 78(4), 493-506.

Ghitany, M., Al-Mutairi, D., Balakrishnan, N., & Al-Enezi, I. (2013). Power Lindley distribution and associated inference. Computational Statistics and Data Analysis, 64, 20-33.

Jorgensen, B. (1982). Statistical properties of the generalized inverse Gaussian distribution. New York: Springer-Verlag.

Lindley, D. V. (1958). Fiducial distributions and Bayes theorem. Journal of the Royal Statistical Society Series B, 20(1), 102-107.

Mudholkar, G., & Srivastava, D. (1993), Exponentiated weibull family for analyzing bathtub failure rate data. IEEE Transactions on Reliability, 42, 299-302.

Nadarajah, S., Bakouch, H. S., & Tahmasbi, R. (2011). A generalized Lindley distribution. Sankhya B, 73(2), 331-359.

Shaked, M., & Shanthikumar, J. G. (1994). Stochastic orders and their applications. New York: Academic Press.

Sharma, V., Singh, S., Singh, U., & Agiwal, V. (2015). The inverse Lindley distribution- A stress-strength reliability model with applications to head and neck cancer data. Journal of Industrial and Production Engineering, 32(3), 162-173.

Sharma, V., Singh, S., Singh, U., & Merovci, F. (2016). The generalized inverse Lindley distribution: A new inverse statistical model for the study of upside-down bathtub survival data. Commun Stat Theory Methods, 45(19), 5709-5729.

Xie, M., Goh, T., & Tang, Y. (2002). A modified weibull extension with bathtub-shaped failure rate function. Reliability Engineering and Systems Safety, 76, 279-285.

Xie, M., & Lai, C. (1996), Reliability analysis using an additive weibull model with bathtub shaped failure rate function. Reliability Engineering and Systems Safety, 52, 87-93.

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