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Identity Theorem in Complex Analysis

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***Abstract:*Let D be an open connected domain in a set of complex number ℂ. Let φ be an analytic complex valued function on open connected domain D. In this paper we are going to accommodate "Identity Theorem" for complex valued function. This paper represents that how an analytic complex valued function becomes identically zero in open connected domain of complex field. However, we will also see consequences of an identity theorem.**

***Index:* Analytic function, Identity, Holomorphic, Limit point, Open connected domain.**

I. INTRODUCTION

In this research article we will discuss about Identity theorem of complex valued function defined on open connected domain. Its statement is that" a complex valued function which is an analytic in open connected domain that contains a point which is limit point of set of zeros of function, then function is identically zero in open connected domain of complex field." It very surprising to us that trigonometric identities which are hold for real number system also hold in complex field via identity theorem (Adreescu & Andrica, 2006).

This concept is one of the best theorem from zeros of function of an analytic function on open connected domain of a complex field (Stein & Shakarchi, 2002).

II. PROBLEM/STRUCTURE

Firstly .we will try to understand what is zeros of a function and poles of a function in complex field. Zeros of a function and poles of a function in a complex field are very much familiar. Further, Zeros of a function are going to be most important foundation to understand concept clearly.

Let φ be analytic complex valued functions in open connected domain D and z_0 is a zeros of φ , i.e., $\varphi(z_0)$=0,then by using Taylor series expansion of a complex valued function φ around a point z_0 it follows that

$$\varphi(z) = (z - z_0)h(z)$$

in a neighbourhood of point z_0,where h is an analytic function in a neighbourhood of a point z_0.

We have some important definitions regarding a zeros of a function are as follows,

Definition-1: Let φ be analytic complex valued functions in open connected domain D and $z_0 \in D$ is called zero of a complex valued function φ of order n if $\varphi^{i}(z_0)$=0,for i=1,2,3…….,(n-1) and $\varphi^{n}(z_0) \neq 0$. We will try to elaborate via examples.

Consider a polynomial $\varphi(z) = z^3+8=(z+2)(z^2-2z+4)$ has a zero of order n=1 at $z_0 = -2$ since $\varphi(z) = (z+2)h(z)$ where $h(z)= (z^2-2z+4)$ and φ(z), h(z)are entire functions and h(-2)=12$\neq 0$.

Note that $z_0 = -2$ is a zero of order n=1 of function φ

Also follows from observation that φ is an analytic complex valued function and $\varphi(-2)=0$ and $\varphi'(-2) \neq 0$(Nair, 2008).

Consider a polynomial $\varphi(z) = (z-1)^2(z+1)$ has a zero of order n=2 at $z_0 = 1$ since $\varphi(z) = (z-1)^2h(z)$

Where $h(z) = (z+1)$ and φ(z), h(z)are entire functions and h(1)=2$\neq 0$.

Note that $z_0 = 1$ is a zero of order n=2 of function φ

Also follows from observation that φ is an analytic.

Complex valued function and $\varphi(1) = 0$, $\varphi'(1) = 0$ & $\varphi''(1) \neq 0$.

(here $\varphi(z) = (z-1)^2(z+1)=(z^3 - z^2 - z + 1)$

So ,$\varphi'(z) = (3z^2 - 2z - 1)$ which implies that $\varphi''(z) = (6z - 2)$ and hence $\varphi''(1) = 4$.

Definition-2: A point $z_0 \in D$ is called as zero of a complex valued function φ of finite order if it is a zero of function φ of order n for some n∈ℕ.

Consider a polynomial $\varphi(z) = (z-1)^3(z+1)$ has a zero of order n=3 at $z_0 = 1$ since $\varphi(z) = (z-1)^3h(z)$

Where $h(z) = (z+1)$ and φ(z), h(z)are entire functions and h(1)=2$\neq 0$.

Note that $z_0 = 1$ is a zero of order n=3 of function φ

Also follows from observation that φ is an analytic.

Complex valued function and $\varphi(1) = 0$, $\varphi'(1) = 0$,

$\varphi''(1) = 0$ & $\varphi'''(1) \neq 0$.

Here $\varphi(z) = (z-1)^3(z+1)$=$(z^4 - 2z^3 + 2z - 1)$So,$\varphi'(z) = (4z^3 - 6z^2 + 2)$ which gives $\varphi''(z)$= $(12z^2 - 12z)$ which implies that $\varphi'''(z) = (24z - 12)$ and hence $\varphi'''(1) = 12$(Brawn & Churchill, 2009).

Consider a complex valued function defined on connected domain D by $\varphi(z) = (e^z\text{-}1)z$.

Here, function φ has a zero of order n=2 at point $z_0 = 0$

Since , $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(0) \neq 0$.

Note that $\varphi(z) = (e^z\text{-}1)z$=$(ze^z\text{-}z)$ which implies that first order derivative of φ is $\varphi'(z) = (ze^z + e^z - 1)$

and second order derivative of function φ is

$\varphi''(z) = (ze^z + 2e^z - 2)$ and hence $\varphi''(0) \neq 0$.

Definition-3:A zero of complex valued function φ which is not of finite order is said to be zero of φ of infinite order.

Now we see concept of pole of complex valued function.

A pole of φ of analytic function is a zero of $\frac{1}{\varphi}$.

Suppose φ_1 and φ_2 be two analytic functions at a point $z_0 \in$ D such that $\varphi_1(z_0) \neq 0$ and φ_2 has a zero of order n at $z_0 \in$ D then quotient $\frac{\varphi_1}{\varphi_2}$ has a pole of order n at z_0.

Hint: define $\varphi_2(z)$=$(z - z_0)^n$ h(z) where $h(z)$ is analytic function and $h(z_0) \neq 0$ and this help us to write quotient $\frac{\varphi_1(z)}{\varphi_2(z)}$.

An identity theorem for any complex valued function stated differently (Kasana, 2005). We will see it.

Statement: let φ be analytic complex valued functions in open connected domain D (Ponnusamy & Silverman, 2006). Let S be set of all zeros of function φ which has a limit point in D, then $\varphi(z) \equiv 0$, for all z∈ D(i.e.,φ(z) is identically zero in D(Lars, 1979).

A. *Example*

1. let φ be entire complex valued functions(SCHAUM's Outlines,2009) in open connected domain D and $\varphi(\frac{1}{3^n})$=0, for all n∈ ℕ.let S={z/φ(z) = 0} be set of all zeros.this implies that "0" is a limit point of S and 0∈ D ⊆ℂ.Therefore, by identity theorem $\varphi(z) \equiv 0$, for all z ∈ D(i.e.φ(z) is identically zero in D).
(Shashtri , 2010).

2. Let g be an entire function in open connected domain D and $g(\frac{1}{5n})$=0, for all naturals n.

Let T={z/g(z) = 0} be set of all zeros.this implies that "0" is a limit point of S and 0∈ D ⊆ℂ.Therefore, by identity theorem $g(z) \equiv 0$, for all z ∈ D(i.e.,g(z) is identically zero in D).(Gamelin, 2001).

III. COROLLARY

A. If φ_1 and φ_2 are two analytic complex valued functions in open connected domain D. if $\varphi_1(z)$=$\varphi_2(z)$ on set S which has limit point(Sarason, 2007) in D, then $\varphi_1(z) \equiv \varphi_2(z)$ on domain D (means φ_1 and φ_2 are identical on domain D). (1)

Proof: - Define $\varphi(z)$=$\varphi_1(z)$-$\varphi_2(z)$, for all z∈ D.As φ_1 and φ_2 are an analytic on D. this implies that $\varphi_1 - \varphi_2$ is

also analytic on domain D. Let S be set of all zeros of φ(z) which has a limit point in D. Therefore, by identity.

theorem, $\varphi(z) \equiv 0$, for all z∈ D.this implies, $\varphi_1(z)$-$\varphi_2(z) \equiv 0$, for all z∈ D.(Rudin, 1987)

So, finally concluded as $\varphi_1(z) \equiv \varphi_2(z)$, for all z∈ D.

B. Let D be an open connected non –empty which is symmetric with respect to the X-axis i.e., z∈D

iff $\bar{z}$ ∈D.suppose φ_1 is holomorphic on D such that it is real on D∩ ℝ,then,

$\varphi_1(\bar{z}) = \overline{\varphi_1(z)}$ (2)

(Conway, 1978)

Where φ_1 is holomorphic function refer to (2).

Hint:-Define $\varphi_2(z)$ =$\overline{\varphi_1(z)}$ and use (2).

Let φ_1 analytic complex valued functions in {z∈ D/|z| <1} and $\varphi_1(\frac{1}{2n+1})$=$\frac{1}{2n+1}$, for all n∈ ℕ

and $\varphi_2\left(\frac{1}{2n+1}\right) = 0$, for all n∈ ℕ ,then by from (1),

$\varphi_1(z) = z$ and $\varphi_2(z)$=0 for all z (Shirali & Vasudeva, 2011).

Now we will show that there is no analytic function φ on connected domain {z∈ D/|z| <1}

Satisfying$\varphi(\frac{1}{n})$=$\frac{(-1)^n}{n^2}$,for all n∈ ℕ . suppose possible there is an analytic function φ which satisfying the above conditions ,then we have following.

$\varphi(\frac{1}{3n})$=$\frac{1}{(3n)^2}$ and $\varphi(\frac{1}{3n-1})$=$\frac{-1}{(3n-1)^2}$, for all n∈ ℕ. (Cufi & Bruna, 2010)

Then, by (1) ,we received $\varphi(z) = z^2$ and

$\varphi(z) = -z^2$ for all z in domain D, which is not possible.

C. Let h analytic complex valued functions in {z∈ D/|z| <1} and $h(\frac{1}{5n+1})$=$\frac{1}{5n+1}$, for all n∈ ℕ and $g\left(\frac{1}{5n+1}\right) = 0$, for all n∈ ℕ ,then by from (1),

$h(z) = z$ and g(z)=0 for all z∈ D.

D. There is no analytic function φ on connected domain {z∈ D/|z| <1}

Satisfying $j(\frac{1}{n})=\frac{(-1)^n}{n^2}$,for all n∈ ℕ . Suppose possible there is an analytic function φ which satisfying.

The above conditions, then we have following

$j(\frac{1}{5n})=\frac{1}{(5n)^2}$ and $j(\frac{1}{5n-1})=\frac{-1}{(5n-1)^2}$, for all n∈ ℕ.

then by (1) ,we received $j(z) = z^2$ and $j(z) = -z^2$ for all z in domain D, which is not possible (Kaur, 2018).

E. Let $\varphi_1:\mathbb{C} \to \mathbb{C}$ be an entire function and let $\varphi_2:\mathbb{C} \to \mathbb{C}$ be defined by $\varphi_2(z) = \varphi_1(z) - \varphi_1(z+1)$,for all z∈ ℂ,then we have
   1. If $\varphi_1\left(\frac{1}{n}\right) = 0$, for all n∈ ℕ ,then φ_1 is a constant function.
   2. If $\varphi_1\left(\frac{1}{n}\right) = \varphi_1\left(\frac{1}{n}+1\right)$, for all n∈ ℕ ,then φ_2 is a constant function.

F. Zeros of analytic functions are isolated.

IV. RESULTS

We have following results from identity theorem for a complex valued function which is analytic on open.

Connected domain is as follows:-

A. Let φ_1 and φ_2 are two analytic complex valued functions in open connected (Alpay, 2015). Domain D. let z_n be a sequence of points having a limit point in D. if $\varphi_1(z_n) \equiv \varphi_2(z_n)$, for each n∈ ℕ ,then $\varphi_1(z) \equiv \varphi_2(z)$, for all z∈ D.

B. Trigonometric identities

We know that $sin(z)$ and $cos(z)$ are entire functions that gives $sin^2(\theta) + cos^2(\theta) = 1$, for all real θ ,then necessarily we get over complex field

$\sin^2(z) + \cos^2(z) = 1$, for all z∈ℂ. (3)

This follows from (1) by taking $\varphi_1(z) = \sin^2(z) + \cos^2(z)$ and $\varphi_2(z)$=1 , for all z∈ D.

This gives us $\cos^2(z)$=1-$\sin^2(z)$, for all z∈ ℂ.

Also, $\sin^2(z)$=1-$\cos^2(z)$, for all z∈ℂ.

(Howie, 2003).

1+$\tan^2$ (z) = $\sec^2$(z), for all z∈ℂ . (4)

This implies 1= $\sec^2$(z)-$\tan^2$ (z), for all z∈ℂ .

Also $\tan^2$ (z) =$\sec^2$(z)-1 ,for all z∈ ℂ .

1+$\cot^2(z) = \text{cosec}^2(z)$, for all z∈ ℂ. (5)

This contributes 1=$\text{cosec}^2(z) - \cot^2(z)$, for all z∈ ℂ.

Also $\cot^2(z) = \text{cosec}^2(z) - 1$,for all z∈ ℂ.

The above trigonometric identities hold provided expressions are holomorphic in ℂ.(Fisher, 1999)

Note that in above trigonometric identities expression on LHS and RHS of the equality sign are:

holomorphic in ℂ(Remmert, 1989).

$Sin(z)$ and $cos(z)$ are trigonometric function refer to (3).

$tan(z)$ and $sec(z)$ are trigonometric function refer to (4).

$cot(z)$ and $cosec(z)$ are trigonometric function refer to (5).

C. Suppose φ is a holomorphic function in a connected domain D that vanishes on a sequence of distinct points with a limit point in D then φ is identically 0.
In other sense, if zeros of a holomorphic function φ in the connected domain D accumulate in D, then φ is identically 0.

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