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Bayesian Analysis of Hjorth Distribution under Generalised Type I Progressive Hybrid Censoring Scheme

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Abstract: **The paper presents analytical and empirical posterior analysis for the two parameter Hjorth distribution under Generalised Type-I Progressive Hybrid Censoring. Its various distributional properties are also derived. Maximum likelihood, asymptotic confidence interval and Bayes estimates are developed for the unknown parameters assuming squared error loss. E-Bayesian and hierarchical Bayesian inferential analyses are also conducted. A simulation study illustrates the theoretic findings in context of the considered censoring, on one classical approach and three Bayes methodologies developed in this paper. Two real data sets are used to demonstrate the model applicability.**

Index Terms: Generalised Progressive Hybrid Censoring, Hjorth Distribution, Bayesian Parametric Estimation, E-Bayes, Hierarchical Bayes Estimate.

I. INTRODUCTION

A *three parameter* Hjorth distribution H(α,β,λ) was introduced by Hjorth (1980) as an alternative lifetime model to the widely accepted Weibull, Rayleigh and exponential distributions . It exhibited increasing, decreasing, constant and bathtub shaped failure rates (IDB). The quest for a new reliability model was motivated by a desire to represent mixture of a set of increasing failure rate (IFR) distributions with a single model. H(α,β,λ) has a competing risk interpretation. It represents lifetime of a mixture of mechanical units, where each unit follows linear failure rate subject to wear out from the beginning of their lifetimes. Its probability density function (*pdf*) is given by $f(x)=\frac{\alpha+\beta x(\lambda x+1)}{(1+\lambda x)^{(\alpha/\lambda)+1}}e^{-\frac{\beta}{2}x^2}; x>0, \alpha\geq 0,\ \beta>0$. Since then this distribution has received very little attention in the context of lifetime studies. Some of its inferential aspects have been studied by Guess *et al*. (1998), such as derivation of conditional densities which characterize the relation between the failure rate and the mean residual life. More recently, Bayes parametric and survival estimation of the *two parameter* Hjorth distribution H(α,β) under progressively type-II censored data has been studied by Yadav *et al*. (2019), by regarding the second shape parameter as fixed (λ=1). So far several of its mathematical and statistical aspects have not been explored in detail. Owing to its IDB nature Hjorth can be seen as a strong reliability-model candidate. The present paper is an effort in this direction. We focus on its mathematical properties and parametric estimators under progressive censoring which is an ideal testing situation for robust product desired in the current markets.

The *pdf* H(α,β) is given by $f(x)=\frac{\alpha+\beta x(x+1)}{(1+x)^{\alpha+1}}e^{-\frac{\beta}{2}x^2}; x>0, \alpha\geq 0, \beta>0$, which is obtained by regarding the additional shape parameter λ as 1 in H(α,β,λ). The corresponding cumulative density function (cdf) is given as $F(x)=1-\frac{e^{-\frac{\beta}{2}x^2}}{(1+x)^{\alpha}}$. The survival function is $S(x)=\frac{e^{-\frac{\beta}{2}x^2}}{(1+x)^{\alpha}}$ which appears as the product of survival function of Rayleigh, $e^{-\frac{\beta}{2}x^2}$, and the survival function of Lomax, $(1+x)^{-\alpha}$, distributions, thereby highlighting the competing risk aspect of the distribution. Hazard function $h(x)=\beta x+\frac{\alpha}{(1+x)}$, is seen as the sum of an increasing and a decreasing term.

The present paper undertakes comparative study of the classical and Bayesian estimators for the two parameter H(α,β) distribution. Section 2 presents some statistical and reliability results for the H(α,β) distribution. Section 3 describes Generalised Type-I Progressive Hybrid Censoring (GT-IPH) mechanism. Section 4 is devoted to the development of Maximum Likelihood Estimates (MLE*s*) under GT-IPH. Section 5 describes construction of Asymptotic Confidence Interval

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(ACI) for the parameter and the reliability estimates. Section 6 traces development of the classical Bayes estimators. Section 7 details E-Bayes estimation for GT-IPH censored sample. Section 8 focuses on the development of Hierarchical Bayes estimates for the model parameters. Simulation study is undertaken in section 9. Model fit is illustrated on two real data sets from the classical literature on reliability. Section 10 concludes the findings of the present study.

II. SOME DISTRIBUTIONAL PROPERTIES

Some statistical properties of the H(α,β) are presented as under,

(i) Mean $= \int_0^\infty x \frac{\alpha+\beta x(x+1)}{(1+x)^{\alpha+1}} e^{-\frac{\beta}{2}x^2}\, dx$

(ii) Median = $\frac{-\alpha+\sqrt{(\alpha^2+2(\beta-\alpha)\log 2)}}{(\beta-\alpha)}$

(iii) Mode is given by the root of the polynomial,

$$\beta^2x^4 - 2\beta^2x^3 + \beta(1-2\alpha-\beta)x^2 + 2\beta(1-\alpha)x + \beta - \alpha(1+\alpha) = 0$$

(iv) Mean Residual Life = $\frac{\int_x^\infty \frac{e^{-\frac{\beta}{2}x^2}}{(1+x)^\alpha}dx}{\frac{e^{-\frac{\beta}{2}x^2}}{(1+x)^\alpha}}$

(v) The r^{th} raw moment $= \int_0^\infty x^r \frac{\alpha+\beta x(x+1)}{(1+x)^{\alpha+1}} e^{-\frac{\beta}{2}x^2}\, dx$

(vi) The r^{th} central moment $= \int_0^\infty (x-\mu_1')^r \frac{\alpha+\beta x(x+1)}{(1+x)^{\alpha+1}} e^{-\frac{\beta}{2}x^2}\, dx$

(vii) Moment generating function $M_X(t) = \int_0^\infty e^{xt} \frac{\alpha+\beta x(x+1)}{(1+x)^{\alpha+1}} e^{-\frac{\beta}{2}x^2}\, dx$

(viii) Characteristic function $= \int_0^\infty e^{ixt} \frac{\alpha+\beta x(x+1)}{(1+x)^{\alpha+1}} e^{-\frac{\beta}{2}x^2}\, dx$

(ix) Quantile function $x_q = \frac{-\alpha+\sqrt{\left(\alpha^2-2(\beta-\alpha)log(1-u)\right)}}{(\beta-\alpha)}$

(x) Uncertainty or randomness measured as Renyi entropy $I_R(\delta) = \frac{\log\{\int_{-\infty}^{\infty} f(x)^\delta dx\}}{1-\delta}, \delta > 0, \delta \neq 1$ and as Shannon entropy $S = E[-\log f(x)]$ is respectively given as , $I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^\infty \frac{\{\alpha+\beta x(x+1)\}^\delta}{(1+x)^{\delta(\alpha+1)}} e^{-\frac{\delta\beta}{2}x^2}\right]$ and

$$S = (\alpha+1)E[\log(1+x)] + \frac{\beta}{2}E[x^2] - E[\log\{\alpha + \beta x(x+1)\}]$$

Graphical representation of density function, distribution function, survival function and hazard function is respectively given ii Fig. 1- Fig. 4.

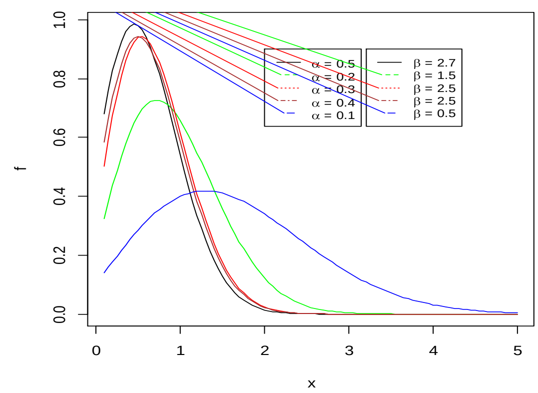

Fig. 1: Plot of *pdf* $f(x)$ of H(α, β) for different combinations of parameter values

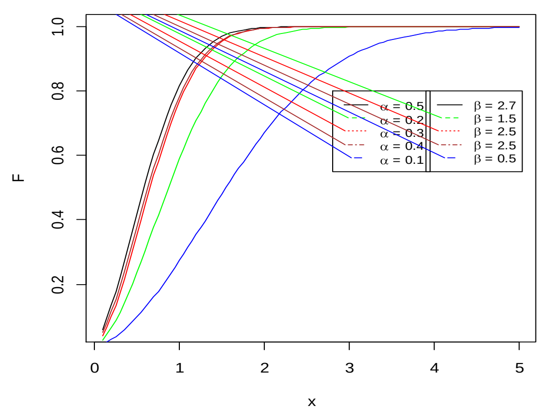

Fig. 2: Plot of *cdf* $F(x)$ of H(α, β) for different combinations of parameter values

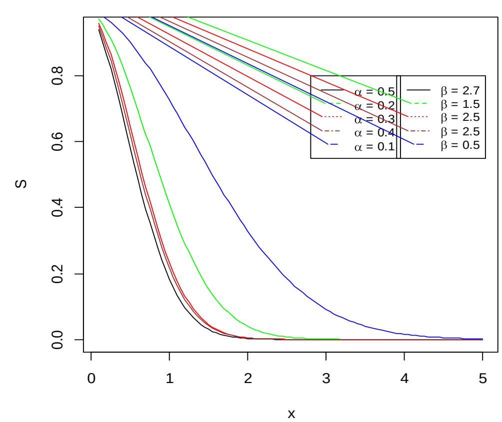

Fig. 3: Plot of survival function $S(x)$ of H(α, β) for different combinations of parameter values

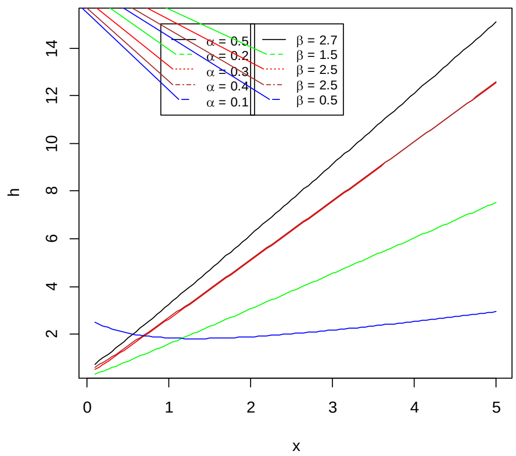

Fig. 4: Plot of hazard function $h(x)$ of $H(\alpha, \beta)$ for different combinations of parameter values

III. GENERALISED TYPE I PROGRESSIVEHYBRID CENSORING SCHEME

Censoring is conducted in life testing experiments to optimize the cost and observe the time constraints. Let $(X_{1:m:n}, \ldots, X_{m:m:n})$ denote a progressive Type-II censored sample, with progressive removals fixed as $(R_1, \ldots, R_m)$. An integer $k(< m)$ and a time $T \in (0, \infty)$ are also pre-decided for determining termination or end time of the life-test experiment which is governed by $\text{Max}\{X_{k:m:n}, \text{Min}\{X_{m:m:n}, T\}\}$. All remaining alive and functioning units are removed at the life-test's end point. This procedure is given by Cho *et al.* (2015) and is referred to as GT-IPH scheme. Let d denote the number of observed failures till time T. Random sample observed under GT-IPH mechanism is schematically represented as follows:

Case I : $\{X_{1:m:n}, \ldots, X_{k:m:n}\}$, if $T < X_{k:m:n} < X_{m:m:n}$

Case II : $\{X_{1:m:n}, \ldots, X_{k:m:n}, \ldots, X_{d:m:n}\}$, if $X_{k:m:n} < T < X_{m:m:n}$

Case III : $\{X_{1:m:n}, \ldots, X_{m:m:n}\}$, if $X_{k:m:n} < X_{m:m:n} < T$

This censoring scheme ensures at least k failures within a moderate time frame thereby overcoming drawbacks of both Type II censoring (longer time frame) and hybrid censoring (too few observed failures) while simultaneously allowing intermittent live removals during the course of trial runs. Also, sample under Case III above, is the conventional progressive Type-II censored sample. We, therefore, develop estimation methods only for Cases I and II.

IV. MAXIMUM LIKELIHOOD ESTIMATION

Following the notations suggested by Seo and Kim (2017), we write the corresponding likelihood functions:

$$L_I(\alpha, \beta) = \left[\prod_{i=1}^{k} \sum_{j=i}^{m} (R_j + 1)\right] f(x_{k:m:n})[1 - F(x_{k:m:n})]^{R_k^*} \prod_{i=1}^{k-1} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{R_i}$$

$$L_{II}(\alpha, \beta) = \left[\prod_{i=1}^{d} \sum_{j=i}^{m} (R_j + 1)\right] [1 - F(x_{k:m:n})]^{R_{d+1}^*} \prod_{i=1}^{d} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{R_i}$$

where $R_k^* = n - k - \sum_{i=1}^{k-1} R_i$ and $R_{d+1}^* = n - d - \sum_{i=1}^{d} R_i$ .The above likelihood functions are expressed in the following compact form,

$$L(\alpha, \beta) = [\prod_{i=1}^{N} \sum_{j=i}^{m} (R_j + 1)][1 - F(Z)]^R \prod_{i=1}^{N} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{R_i} \quad (1)$$

such that $N = \begin{cases} k & \text{for case I} \\ d & \text{for case II} \end{cases}$, $Z = \begin{cases} X_{k:m:n} & \text{for case I} \\ T & \text{for case II} \end{cases}$, and $R = \begin{cases} R_k^* - R_k & \text{for case I} \\ R_{d+1}^* & \text{for case II} \end{cases}$

Using (1), likelihood function of H(α,β) is given by

$$L(\alpha, \beta) = \left[\prod_{i=1}^{N} \sum_{j=i}^{m} (R_j + 1)\right] \left[\frac{e^{-\frac{\beta}{2} Z^2}}{(1+Z)^{\alpha}}\right]^R \prod_{i=1}^{N} \frac{\alpha + \beta x_{i:m:n}(x_{i:m:n} + 1)}{(1 + x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2} x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2} x_{i:m:n}^2}}{(1 + x_{i:m:n})^{\alpha}}\right]^{R_i}$$

The corresponding log-likelihood function is:

$$l(\alpha, \beta) = C - \frac{\beta}{2}[RZ^2 + \sum_{i=1}^{N}(R_i + 1)x_{i:m:n}^2] - \alpha \log(1 + Z) + \sum_{i=1}^{N} \log\{\alpha + \beta x_{i:m:n}(x_{i:m:n} + 1)\} - \sum_{i=1}^{N} \log(1 + x_{i:m:n})\{\alpha(R_i + 1) + 1\} \quad (2)$$

where $C = \log\left[\prod_{i=1}^{N} \sum_{j=i}^{m} (R_j + 1)\right]$. Differentiating with respect to α and β, we obtain

$$\frac{\partial l(\alpha,\beta)}{\partial\alpha} = -Rlog(1+Z) + \sum_{i=1}^{N}\frac{1}{\{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)\}} - \sum_{i=1}^{N}(R_i+1)log(1+x_{i:m:n})$$

$$\frac{\partial l(\alpha,\beta)}{\partial\beta} = -\frac{1}{2}\left[RZ^2 + \sum_{i=1}^{N}(R_i+1)\,x_{i:m:n}^2\right] - \sum_{i=1}^{N}\frac{x_{i:m:n}(x_{i:m:n}+1)}{\{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)\}}$$

The above equations do not appear in closed form, therefore, we obtain MLE $(\hat{\alpha}_M,\hat{\beta}_M)$ of the unknown parameters (α, β) through the use of Newton Raphson (N-R) iterative numerical approximation method.

V. ASYMPTOTIC CONFIDENCE INTERVAL

Fisher's information matrix is used for constructing the asymptotic variance-covariance matrix $M(\hat{\alpha},\hat{\beta})$ and 100(1-α) % ACI for the unknown parameters to be estimated.

$$M(\hat{\alpha},\hat{\beta}) = \left[E\begin{pmatrix}\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha^2} & \frac{\partial^2 l(\alpha,\beta)}{\partial\alpha\partial\beta}\\ \frac{\partial^2 l(\alpha,\beta)}{\partial\beta\partial\alpha} & \frac{\partial^2 l(\alpha,\beta)}{\partial\beta^2}\end{pmatrix}\right]^{-1} = \begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta}\\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta}\end{pmatrix}$$

Therefore, by using the concept of large sample theory, the 100(1-α)% ACIs for the unknown parameters α and β are given by $[\hat{\alpha}_L,\hat{\alpha}_U] = \hat{\alpha} \pm z_{\alpha/2}\sqrt{\hat{V}_{\alpha\alpha}}$ and $[\hat{\beta}_L,\hat{\beta}_U] = \hat{\alpha} \pm z_{\alpha/2}\sqrt{\hat{V}_{\beta\beta}}$ (2)

respectively, where $Z_{\alpha/2}$ is the critical value of the standard normal variate and

$$\hat{V}_{\alpha\alpha} = -\frac{1}{E\left[\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha^2}\right]},\quad \hat{V}_{\alpha\beta} = -\frac{1}{E\left[\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha\partial\beta}\right]},\quad \hat{V}_{\beta\alpha} = -\frac{1}{E\left[\frac{\partial^2 l(\alpha,\beta)}{\partial\beta\partial\alpha}\right]},\quad \hat{V}_{\beta\beta} = -\frac{1}{E\left[\frac{\partial^2 l(\alpha,\beta)}{\partial\beta^2}\right]}.$$

Additionally, $\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha^2} = -\frac{1}{\sum_{i=1}^{N}\{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)\}^2}$, $\frac{\partial^2 l(\alpha,\beta)}{\partial\beta^2} = -\sum_{i=1}^{N}\frac{\{x_{i:m:n}(x_{i:m:n}+1)^2\}}{\{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)\}^2}$ and

$$\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha\partial\beta} = \frac{\partial^2 l(\alpha,\beta)}{\partial\beta\partial\alpha} = -\sum_{i=1}^{N}\frac{x_{i:m:n}(x_{i:m:n}+1)}{\{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)\}^2}.$$

VI. BAYESIAN ESTIMATION

Assuming that the prior distribution for shape parameter α follows Gamma(a, b) and for the scale parameter β follows Gamma(c,d), the joint prior distribution of the unknown parameters α and β is given as $\pi(\alpha,\beta) = \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}}{\Gamma\alpha\Gamma c}$. From the likelihood function $L(\alpha,\beta)$ and the joint prior density function $\pi(\alpha,\beta)$, the joint posterior density function is given by

$$p(\alpha,\beta|X) = \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}d\alpha d\beta}$$

The marginal posterior density function for the unknown parameter α is given as,

$$p(\alpha|\beta,X) = \int p(\alpha,\beta|X)d\beta = \int\frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}d\alpha d\beta}d\beta$$

Therefore, the Bayesian estimator of the unknown parameter α under the squared error loss function (SELF) which is the posterior mean, is expressed as,

$$\hat{\alpha}_B = \int \alpha p(\alpha|\beta,X)d\alpha = \int\alpha\int\frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}d\alpha d\beta}d\beta$$

(3)

The marginal posterior density function for the unknown parameter β is given as,

$$p(\beta|\alpha,X)=\int p(\alpha,\beta|X)d\alpha$$

$$=\int \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}d\alpha d\beta}d\alpha$$

Therefore, the Bayesian estimator of the unknown parameter β under SELF which is the posterior mean, is expressed as,

$$\hat{\beta}_B=\int \beta p(\beta|\alpha,X)d\beta$$

$$=\int\beta\int \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^{R}\prod_{i=1}^{N}\frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}}e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}d\alpha d\beta}d\alpha\, d\beta$$

(4)

In the above equations the integrals are not obtained in closed form therefore, we obtain Bayes estimators ($\hat{\alpha}_B$, $\hat{\beta}_B$) of the unknown parameters (α, β) using Markov Chain Monte Carlo (MCMC) iterative approximation technique.

VII. E-Bayesian Estimation

E-Bayesian paradigm proposed by Han (2009) for failure rate estimation of small samples arising from test trials on robust items which are obtained under censoring or truncation, is regarded as an efficient alternative to the classical and the conventional Bayesian perspective in recent times. E-Bayes estimate is popularly referred to as Expected Bayes or Extended Bayes estimate. Assuming that α ~ Gamma(a, b) and β ~ Gamma(c, d), we restrict the hyper parameters as $0 < a < 1$, $b > 0$ and $0 < c < 1$, $d > 0$ in order to ensure the defining condition that the prior distribution should be decreasing function of the random variable for which it is specified. This is ensured as, the first derivative of the respective prior distribution with respect to α and β respectively are given as $\frac{d\pi_1(\alpha|a,b)}{d\alpha}=\frac{b^a\alpha^{a-1}e^{-b\alpha}}{\Gamma a}[(a-1)-b\alpha]$ and $\frac{d\pi_2(\beta|c,d)}{d\beta}=\frac{d^c\beta^{c-1}e^{-d\beta}}{\Gamma c}[(c-1)-d\beta]$. Also, the larger the hyper parameters b and d are, the thinner is the tail of gamma (decreasing) function. Berger (1985) has argued that the thinner tailed prior distributions often reduce the robustness of Bayesian estimates. Accordingly, b and d should be bounded above by some s and t respectively, where $s > 0$ and $t > 0$ are fixed scalars to be determined for individual trial run. The rescaled boundaries are given as $0 < a < 1$, $0 < b < s$ and $0 < c < 1$, $0 < d < t$ respectively. Thus, defining the (continuous) finite E-Bayes parameters $\hat{\alpha}_{EB}(a,b)$ and $\hat{\beta}_{EB}(c,d)$, we write

$$\hat{\alpha}_{EB}=E[\hat{\alpha}_B(a,b)]=\iint \hat{\alpha}_B(a,b)\,\mathrm{Gamma}(a,b)\,da\,db \tag{5}$$

and

$$\hat{\beta}_{EB}=E\left[\hat{\beta}_B(c,d)\right]=\iint \hat{\beta}_B(c,d)\mathrm{Gamma}(c,d)\,dc\,dd \tag{6}$$

Postulating three different prior specifications for the underlying hyper parameters, such that they are increasing, constant and decreasing functions of the hyper parameter respectively,

$$\pi_{11}(a,b)=\frac{2(s-b)}{s^2}\ ;\ 1<b<s$$

$$\pi_{12}(a,b)=\frac{1}{s}\ ;\ 1<b<s$$

$$\pi_{13}(a,b)=\frac{2b}{s^2}\ ;\ 1<b<s \qquad (7)\text{ - }(9)$$

and

$$\pi_{21}(c,d)=\frac{2(t-d)}{t^2}\ ;\ 1<d<t$$

$$\pi_{22}(c,d)=\frac{1}{t}\ ;\ 1<d<t$$

$$\pi_{23}(c,d)=\frac{2d}{t^2}\ ;\ 1<d<t \qquad (10)\text{ -}(12)$$

Let $X_{1:m:n}, \dots, X_{N:m:n}$ represent the sample observations from the distribution H(α,β). Then E-Bayes estimate of the unknown parameters α and β under SELF,

(i) under π_{11} and π_{21} are respectively given as (13) and (14) as under,

$$\hat{\alpha}_{EB1} = \iint \hat{\alpha}_B(a,b)\pi_{11}(a,b)\, dadb$$

$$= \iint \left[\int \alpha \int \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)} \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^R \prod_{i=1}^{N} \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)} \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^R \prod_{i=1}^{N} \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i} d\alpha d\beta} d\beta\, d\alpha \right] \frac{2(s-b)}{s^2} dadb \tag{13}$$

and

$$\hat{\beta}_{EB1} = \iint \hat{\beta}_B(c,d)\pi_{21}(c,d)\, dcdd$$

$$= \iint \left[\int \beta \int \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)} \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^R \prod_{i=1}^{N} \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)} \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^R \prod_{i=1}^{N} \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i} d\alpha d\beta} d\alpha\, d\beta \right] \frac{2(t-d)}{t^2} dcdd \tag{14}$$

(ii) under π_{21} and π_{22} are respectively given as (15) and (16) as under,

$$\hat{\alpha}_{EB2} = \iint \hat{\alpha}_B(a,b)\pi_{12}(a,b)\, dadb$$

$$= \iint \left[\int \alpha \int \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)} \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^R \prod_{i=1}^{N} \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)} \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^{\alpha}}\right]^R \prod_{i=1}^{N} \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^{\alpha}}\right]^{R_i} d\alpha d\beta} d\beta\, d\alpha \right] \frac{1}{s} dadb \tag{15}$$

and

$$\hat{\beta}_{EB2} = \iint \hat{\beta}_B(c,d)\pi_{22}(c,d)\, dcdd$$

$$= \iint \left[\int \beta \int \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} d\alpha d\beta} d\alpha\, d\beta \right] \frac{1}{t}\, dcdd$$

(16)

(iii) under π_{13} and π_{32} are respectively given as (17) and (18) as under,

$$\hat{\alpha}_{EB3} = \iint \hat{\alpha}_B(a,b)\pi_{13}(a,b)\, dadb$$

$$= \iint \left[\int \alpha \int \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} d\alpha d\beta} d\beta\, d\alpha \right] \frac{2b}{s^2}\, dadb$$

(17)

$$\hat{\beta}_{EB3} = \iint \hat{\beta}_B(c,d)\pi_{23}(c,d)\, dcdd$$

$$= \iint \left[\int \beta \int \frac{\alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i}}{\iint \alpha^{a-1}\beta^{c-1}e^{-(b\alpha+d\beta)}\left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2}\left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} d\alpha d\beta} d\alpha\, d\beta \right] \frac{2d}{t^2}\, dcdd$$

(18)

We obtain solution to the above *eqs.* (13)-(18) through the iterative MCMC technique

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VIII. Hierarchical Bayesian Estimation under SELF

Assuming exponential conditional priors $g(\alpha|b) = be^{-b\alpha}$,$b > 0$ under the hyper parameter specifications given by (7)-(9), the unconditional hierarchical priors for the unknown parameter α are elicited as,

$$g_{11}(\alpha) = \frac{2}{s^2}\int_0^s b(s-b)e^{-b\alpha}\, db$$

$$g_{21}(\alpha) = \frac{1}{s}\int_0^s be^{-b\alpha}\, db$$

Now, ***Hierarchical Bayes estimator of α under SELF***

$$g_{31}(\alpha) = \tfrac{2}{s^2}\textstyle\int_0^s b^2 e^{-b\alpha}\, db \qquad (19)\text{-}(21)$$

Similarly, assuming exponential conditional priors $g(\beta|d) = de^{-d\beta}$,$d > 0$ under the hyper parameter specifications given by (10)-(12), the unconditional hierarchical priors for the unknown parameter β are elicited as,

$$g_{12}(\beta) = \tfrac{2}{t^2}\textstyle\int_0^t d(t-d)e^{-d\beta}\, dd$$

$$g_{22}(\beta) = \tfrac{1}{t}\textstyle\int_0^t de^{-d\beta}\, dd$$

$$g_{32}(\beta) = \tfrac{2}{t^2}\textstyle\int_0^t d^2 e^{-d\beta}\, dd \qquad (22)\text{-}(24)$$

(i) with respect to the hierarchical prior $g_{11}(\alpha)$ is $\hat{\alpha}_{HB1} = \frac{\int_0^\infty \alpha L(\alpha,\beta) g_{11}(\alpha) d\alpha}{\int_0^\infty L(\alpha,\beta) g_{11}(\alpha) d\alpha}$. Thus,

$$\hat{\alpha}_{HB1} = \frac{\int_0^\infty \alpha \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{2}{s^2}\int_0^s b(s-b)e^{-b\alpha} db\right) d\alpha}{\int_0^\infty \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{2}{s^2}\int_0^s b(s-b)e^{-b\alpha} db\right) d\alpha}$$

(ii) with respect to the hierarchical prior $g_{21}(\alpha)$ is $\hat{\alpha}_{HB2} = \frac{\int_0^\infty \alpha L(\alpha,\beta) g_{21}(\alpha) d\alpha}{\int_0^\infty L(\alpha,\beta) g_{21}(\alpha) d\alpha}$. Thus,

$$\hat{\alpha}_{HB2} = \frac{\int_0^\infty \alpha \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{1}{s}\int_0^s be^{-b\alpha} db\right) d\alpha}{\int_0^\infty \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{1}{s}\int_0^s be^{-b\alpha} db\right) d\alpha}$$

(iii) with respect to the hierarchical prior $g_{23}(\alpha)$ is $\hat{\alpha}_{HB3} = \frac{\int_0^\infty \alpha L(\alpha,\beta) \pi_{31}(\alpha) d\alpha}{\int_0^\infty L(\alpha,\beta) \pi_{31}(\alpha) d\alpha}$.Thus,

$$\hat{\alpha}_{HB3} = \frac{\int_0^\infty \alpha \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{2}{s^2}\int_0^s b^2e^{-b\alpha} db\right) d\alpha}{\int_0^\infty \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{2}{s^2}\int_0^s b^2e^{-b\alpha} db\right) d\alpha}$$

(25-27)

Now, ***Hierarchical Bayes estimator of β under SELF***

(i) with respect to the hierarchical prior $g_{12}(\beta)$ is $\hat{\beta}_{HB1} = \frac{\int_0^\infty \beta L(\alpha,\beta) \pi_{12}(\beta) d\beta}{\int_0^\infty L(\alpha,\beta) \pi_{12}(\beta) d\beta}$

$$\hat{\beta}_{HB1} = \frac{\int_0^\infty \beta \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{2}{t^2}\int_0^t d(t-d)e^{-d\beta} dd\right) d\beta}{\int_0^\infty \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{2}{t^2}\int_0^t d(t-d)e^{-d\beta} dd\right) d\beta}$$

(ii) with respect to the hierarchical prior $g_{22}(\beta)$ is $\hat{\beta}_{HB2} = \frac{\int_0^\infty \beta L(\alpha,\beta) \pi_{22}(\beta) d\beta}{\int_0^\infty L(\alpha,\beta) \pi_{22}(\beta) d\beta}$

$$\hat{\beta}_{HB2} = \frac{\int_0^\infty \beta \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{1}{t}\int_0^t de^{-d\beta} dd\right) d\beta}{\int_0^\infty \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{1}{t}\int_0^t de^{-d\beta} dd\right) d\beta}$$

(iii) with respect to the hierarchical prior $g_{32}(\beta)$ is $\hat{\beta}_{HB3} = \frac{\int_0^\infty \beta L(\alpha,\beta) \pi_{32}(\beta) d\beta}{\int_0^\infty L(\alpha,\beta) \pi_{32}(\beta) d\beta}$

$$\hat{\beta}_{HB3} == \frac{\int_0^\infty \beta \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{2}{t^2}\int_0^t d^2 e^{-d\beta}\, dd\right) d\beta}{\int_0^\infty \left[\frac{e^{-\frac{\beta}{2}Z^2}}{(1+Z)^\alpha}\right]^R \prod_{i=1}^N \frac{\alpha+\beta x_{i:m:n}(x_{i:m:n}+1)}{(1+x_{i:m:n})^{\alpha+1}} e^{-\frac{\beta}{2}x_{i:m:n}^2} \left[\frac{e^{-\frac{\beta}{2}x_{i:m:n}^2}}{(1+x_{i:m:n})^\alpha}\right]^{R_i} \left(\frac{2}{t^2}\int_0^t d^2 e^{-d\beta}\, dd\right) d\beta}$$

(28)-(30)

It is observed that the above integrals (25)-(30) are not obtained in closed forms, therefore for obtaining hierarchical Bayes estimates of the unknown parameters for the H(α,β) density, we use MCMC method.

IX. SIMULATION

In this section, we compare simulated performance of the ML, Bayes, E-Bayes and hierarchical Bayes estimates of the unknown parameters developed in the earlier sections. MLEs $\hat{\alpha}_M$ and $\hat{\beta}_M$ are computed with the help of N-R method based on 1000 replications. For obtaining Bayes estimators $\hat{\alpha}_B$ and $\hat{\beta}_B$ under SELF the hyper-parameters are assumed as (a, b) = (2, 2) and (c, d) = (2, 3). Sensitivity of E-Bayes and Hierarchical Bayes estimators of parameters α and β is monitored by assigning the following upper bounds to s = 10, s = 50, s = 100; and to t = 20, t = 100, t = 200. MCMC is iterated 10000 times. All simulation is implemented using R codes.

The obtained estimates and their corresponding Mean Square Errors (MSE*s*) are presented in Tables I-II Comparative performance is assessed on the basis of their MSE*s*. It is observed that for fixed *n*, *k*, and *T*, the MSEs are found to take smaller values as *m* is increased. For fixed *n*, *m*, and *T*, the MSE*s* decrease as *k* increases. For fixed *n*, *m*, and *k*, the MSE*s* decrease as *T* rises. Thus, more observed failures and longer trial period leads to improved parameter estimation. In addition, the estimates are not sensitive to the upper bound on the hyper parameter either in case of E–Bayes or in case of Hierarchical Bayes method. Efficiencies (E) of various competitive estimators studied in this paper are presented as under:

$$\mathbf{E_{MLE} < E_{Classical\ Bayes} < E_{E\text{-}Bayes} < E_{Hierarchical\ Bayes}}$$

Table I: Computation of ML $\hat{\alpha}_M$, classical Bayes $\hat{\alpha}_B$, E-Bayes $\hat{\alpha}_{EB1}, \hat{\alpha}_{EB2}, \hat{\alpha}_{EB3}$ and Hierarchical Bayes $\hat{\alpha}_{HB1}, \hat{\alpha}_{HB2}, \hat{\alpha}_{HB3}$ estimators along with their MSEs in brackets.

								s*			s*	
							10	50	100	10	50	100
n	m	N	T	Case	$\hat{\alpha}_M$	$\hat{\alpha}_B$	$\hat{\alpha}_{EB1}$	$\hat{\alpha}_{EB2}$	$\hat{\alpha}_{EB3}$	$\hat{\alpha}_{HB1}$	$\hat{\alpha}_{HB2}$	$\hat{\alpha}_{HB3}$
20	18	4	0.5	I	2.4073	2.4551	2.4502	2.5448	2.4723	2.5514	2.5548	2.5479
					(0.0820)	(0.0537)	(0.0382)	(0.0336)	(0.0290)	(0.0292)	(0.0297)	(0.0287)
				II	2.4460	2.5096	2.5070	2.5068	2.5007	2.5240	2.5337	2.5259
					(0.0743)	(0.0452)	(0.0204)	(0.0202)	(0.0204)	(0.0205)	(0.0245)	(0.0237)
			1.5	I	2.3859	2.4586	2.4693	2.5308	2.4878	2.5499	2.5400	2.5352
					(0.0906)	(0.0523)	(0.0334)	(0.0322)	(0.0301)	(0.0280)	(0.0261)	(0.0248)
				II	2.4475	2.4979	2.5066	2.5041	2.5109	2.5189	2.5271	2.5240
					(0.0872)	(0.0401)	(0.0211)	(0.0271)	(0.0243)	(0.0210)	(0.0222)	(0.0212)
30	26	6	0.5	I	2.3917	2.4660	2.4875	2.5290	2.4709	2.5336	2.5297	2.5202
					(0.0754)	(0.0501)	(0.0305)	(0.0311)	(0.0289)	(0.0302)	(0.0300)	(0.0264)
				II	2.4503	2.4978	2.4966	2.4937	2.4922	2.5179	2.5159	2.5142
					(0.0662)	(0.0424)	(0.0200)	(0.0260)	(0.0203)	(0.0205)	(0.0280)	(0.0200)
			1.5	I	2.4309	2.4767	2.4811	2.5112	2.4852	2.5233	2.5374	2.5187
					(0.0701)	(0.0467)	(0.0288)	(0.0302)	(0.0265)	(0.0298)	(0.0352)	(0.0239)
				II	2.4822	2.4825	2.5090	2.5083	2.5077	2.5198	2.5111	2.5109
					(0.0623)	(0.0421)	(0.0176)	(0.0150)	(0.0188)	(0.0189)	(0.0262)	(0.0104)

s* represents upper bound of the scale hyperparameter

Table II: Computation of ML $\hat{\beta}_M$, classical Bayes $\hat{\beta}_B$, E-Bayes $\hat{\beta}_{EB1}, \hat{\beta}_{EB2}, \hat{\beta}_{EB3}$ and Hierarchical Bayes $\hat{\beta}_{HB1}, \hat{\beta}_{HB2}, \hat{\beta}_{HB3}$ estimators along with their MSEs in brackets.

							t*			t*		
							20	100	200	20	100	200
n	m	N	T	Case	$\hat{\beta}_M$	$\hat{\beta}_B$	$\hat{\beta}_{EB1}$	$\hat{\beta}_{EB2}$	$\hat{\beta}_{EB3}$	$\hat{\beta}_{HB1}$	$\hat{\beta}_{HB2}$	$\hat{\beta}_{HB3}$
20	18	4	0.5	I	2.8054	2.9531	2.9592	2.9515	2.9755	3.2560	3.3818	3.3455
					(0.0960)	(0.0522)	(0.0438)	(0.0404)	(0.0341)	(0.0421)	(0.0496)	(0.0488)
				II	2.9210	2.9734	2.9778	2.9729	2.9877	3.1745	3.2159	3.2237
					(0.0899)	(0.0488)	(0.0351)	(0.0329)	(0.0290)	(0.0379)	(0.0339)	(0.0376)
			1.5	I	2.8844	2.9511	2.9637	2.9645	2.9817	3.2346	3.2544	3.2620
					(0.0881)	(0.0545)	(0.0421)	(0.0430)	(0.0313)	(0.0414)	(0.0482)	(0.0441)
				II	2.9459	2.9834	2.9816	2.9759	2.9967	3.1537	3.1978	3.1782
					(0.0742)	(0.0429)	(0.0355)	(0.0310)	(0.0256)	(0.0372)	(0.0321)	(0.0307)
30	26	6	0.5	I	2.8917	2.9688	2.9488	2.9621	2.9756	3.3001	3.2341	3.2418
					(0.0768)	(0.0490)	(0.0426)	(0.0423)	(0.0332)	(0.0411)	(0.0453)	(0.0436)
				II	2.9518	2.9755	2.9719	2.9981	2.9948	3.2103	3.1848	3.1574
					(0.0657)	(0.0466)	(0.0329)	(0.0389)	(0.0288)	(0.0349)	(0.0310)	(0.0301)
			1.5	I	2.9309	2.9744	2.9728	2.9562	2.9861	3.2091	3.2232	3.1912
					(0.0762)	(0.0431)	(0.0401)	(0.0378)	(0.0322)	(0.0401)	(0.0414)	(0.0429)
				II	2.9877	2.9897	2.5090	2.9817	2.9933	3.1284	3.1276	3.1082
					(0.0600)	(0.0361)	(0.0320)	(0.0299)	(0.0283)	(0.0357)	(0.0300)	(0.0293)

t* represents upper bound of the scale hyperparameter

X. REAL DATA

We consider two real data sets from Lawless (2003) for illustrating application to real life physical situation and portray the model fit to five competitive statistical lifetime models. Data set 1 (*pp*. 98) represents number of million revolutions before failure for each of 23 ball-bearings. These observations arise from test on endurance of deep-groove ball bearings. Lawless (2003) has proposed Log logistic distribution as a suitable model candidate for this data in preference to other lifetime models . On empirical assessment this data set is found to be a positively skew data which makes it a possible candidate for $H(\alpha,\beta)$ distribution.

Data set 2 (*pp*. 112) represents the number of cycles to failure for a group of 60 electrical appliances in a life test. There are a substantial number of small failure times which suggests that the hazard function may be high for small failure times. This fact has motivated us to use the data as a possible candidate for being represented by the $H(\alpha,\beta)$ distribution and in competition with some more popular lifetime models.

Table III: Model fit metrics for data set 1.

Model	Data – 1 $-\log\hat{l}$	AIC	BIC
Hjorth	35.547	821.578	823.849
Lindley	40.315	929.239	931.510
Normal	55.148	1272.410	1274.680
Gamma	73.341	1690.845	1693.116
Log-logistic	82.545	1902.524	1904.795

Table IV: Model fit metrics to data set 2.

Model	Data – 2 $-\log\hat{l}$	AIC	BIC
Hjorth	18.388	426.926	422.771
Lindley	37.062	854.434	858.589
Normal	59.329	1368.559	1370.840
Gamma	24.330	563.597	567.752
Log-logistic	33.617	777.182	781.317

To test the goodness of fit of the above distributions, we have used estimated negative log likelihood function ($-ln$ L), the Akaike information criterion as AIC =-2ln L+2k and Bayesian information criterion as BIC= -2ln L+ kln n , where k is the number of parameters in the distribution, n is the number of observations in the given data set, and L is the maximized value of likelihood function of the estimated model. Best distribution is indicated by the lowest values of the respective -lnL, AIC and BIC statistics. The corresponding values are reported in Table III and Table IV. The two parameter Hjorth model is found to fit most suitably to the chosen data sets.

Thus, we propose $H(\alpha,\beta)$ model as a reliability model in context of robust equipment showing some early failures.

XI. CONCLUSION

The present work is an effort in the direction of continuous exploration for new and better fitted lifetime distributions for machine components and physical equipment. We present mathematical properties and develop expressions for the classical and the Bayes estimators for H(α,β) reliability model under GT-IPH. GT-IPH ensures a certain minimum failure observations while simultaneously limiting the experimental time. Such preconceived group-removal along with an observed failure provides a good test-trial strategy especially in case of robust items. We undertake Bayesian analysis using three non-overlapping methodologies- classical, E-Bayes and hierarchical Bayes. Bayes methodologies are found superior in terms of providing more efficient estimates with respect to the MSE of the estimates as compared to the conventional MLE strategy. Among them hierarchical Bayes estimates appear to be closest to the true parameters closely followed by E– Bayes estimators. We also analyse two real data by several models to get an impression of the sensitivity to model assumptions. Illustrations undertaken in this paper through simulated and real data sets support the candidature of Hjorth distribution as a reliability and life-testing model.

DECLARATION

Authors declare that there is no conflict of interest.

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