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One parameter bathtub model and its real life application

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***Abstract*—The present paper, presents a new single parameter probability distribution having bathtub and increasing hazard rates. It also discusses some of its important statistical properties like order relationship, moments, conditional moments, generating functions, mean deviations, quantile function, median and Shannon entropy. The unknown parameter is estimated by the technique of maximum likelihood estimation. At last, the applicability of the model have been utilised by two different datasets over eight other lifetime models.**

***Index Terms*—Bathtub Hazard rate, Lifetime model, Lindley distribution, Maximum likelihood estimation, Statistical Properties**

I. Introduction

The lifetime distributions are used to analyse the real-life problems, especially in applied field like in engineering, marketing, medical, banking, finance and in others. Since, a lifetime model has its own merits and demerits and can be considerable for a specific area of life. So that, a numbers of lifetime models are proposed in statistical literature as; Weibull, gamma, exponential, Burr, Pareto, etc. In this series, Lindley (1958) proposed a new distribution that was popularised latter by his name and called it as Lindley distribution. The cumulative distribution function (CDF) of the Lindley distribution is defined as,

$$G(x) = 1 - e^{-\theta x}\frac{1+\theta+\theta x}{1+\theta} \qquad \theta, x \geq 0. \tag{1}$$

The nature of the model was increasing hazard type and very applicable to study stress-strength reliability. This property was studied by Ghitany et al. (2008). After that, several authors work on it and proposed various lifetime models based on it as; Ghitany et al. (2008a) proposed Poisson-Lindley distribution, Ghitany and Al-Mutairi (2008b) proposed size biased Poisson Lindley model, Elbatal et al. (2013) proposed three-parameter generalized Lindley distribution, Deniz and Ojeda (2011) proposed discrete Lindley distribution, Nadarajah et al. (2011) proposed two-parameter generalized Lindley distribution, Hassan et al. (2016) proposed Weibull-Quasi Lindley distribution, Rashid and Jan (2016) proposed Lindley power series distribution, Maurya et al. (2017a) proposed a new transformed Lindley distribution based on DUS transformation (Kumar et al. (2015)). Recently, Maurya et al. (2020a) proposed an extended Lindley distribution, and Maurya et al. (2020b) proposed a generalized Lindley distribution. But one common thing appears nearly in all the generalized models that they add up some additional parameters to increase the flexibility of its baseline model. Perhaps taking this point, Maurya et al. (2018) suggest another transformation technique that has more flexibility without adding a parameter. The proposed technique gives the CDF $F(x)$ on the basis of baseline CDF $G(x)$ by the method

$$F(x) = \frac{\log(1+G(x))}{\log 2}. \tag{2}$$

By using this transformation, in this paper, I am introducing a new probability distribution based on Lindley, which also incorporate all the properties of its baseline model along with more flexibility in term of fitting without any additional parameter. Let the random variable (RV) X have the baseline CDF is Lindley distribution, then by using the same concept, the CDF of our proposed distribution is

$$F(x) = \frac{\log\left[2 - e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]}{\log 2} \qquad \theta, x \geq 0 \tag{3}$$

and the associated probability density function (PDF) is:

$$f(x) = \frac{\theta^2}{(\theta+1)\log 2}\frac{(1+x)e^{-\theta x}}{\left[2 - e^{-\theta x}(1+\frac{\theta x}{\theta+1})\right]} \qquad \theta, x \geq 0 \tag{4}$$

and names it as Logarithmic Transformed Lindley (LoTL) distribution.

The whole paper is organised as follows: Section 2, discuss the nature of distribution function and its hazard rate. The Section 3, study some statistical properties, whereas, Section 4, deals the maximum likelihood procedure for the parameter estimation. Section 5, discusses two real datasets to show the model superiority over eight other models. Section 6, concludes the whole paper.

II. Shapes of the Distribution and Its Failure Rate

The graphical representation is a key aspect of every distribution function, as it helps to recognise the nature of its

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density, CDF and types of hazard rate function. The equations (3) and (4) gives the CDF and PDF plots for different values of the parameters θ and it is shown in Figure 1. The PDF plot shows that it goes to zero as the RV tends to infinity and the CDF plot represent that it holds stochastic order relationship as looking the value of CDF in association with X. This can be proved mathematically in subsequent Sections. The hazard rate function gives the change in failure rate over the lifetime of a system. The corresponding hazard rate function is,

$$h(x) = \frac{\theta^2(1+x)e^{-\theta x}}{(\theta+1)\left[2 - e^{-\theta x}(1+\frac{\theta x}{\theta+1})\right]\log\left[\frac{2}{2-e^{-\theta x}(1+\frac{\theta x}{\theta+1})}\right]}. \quad (5)$$

However, the shapes of hazard rate for different choices of the parameter value are plotted in Figure 1. From this, one can see that the proposed LoTL model may have increasing and bathtub hazard rate. Since, we know that the IHR holds the relation of $IHR \Rightarrow IHRA \Rightarrow NBU \Rightarrow NBUE$, where $IHRA$ is increasing hazard rate average, NBU is for new better than used and $NBUE$ for new better than used in expectation (see Barlow and Proschan (1975), Gupta et al. (1998), Marshall and Olkin (2007)). So, the proposed model also hold these relationship.

A. Limiting behaviour of the distribution

The limiting value of PDF of the proposed model can be obtained from the equation (4). Since, the extreme values of X (0 and ∞) as, $\lim_{x\to 0} f(x) = \frac{\theta^2}{(1+\theta)\log 2}$ and $\lim_{x\to\infty} f(x) = 0$. Similarly, the limiting behaviour of hazard function can be known by using the equation (5). As, $\lim_{x\to 0} h(x) = \frac{\theta^2}{(1+\theta)\log 2}$. Thus, one can say that the limiting values of proposed distribution is $\frac{1}{\log 2}$ times value of the Lindley model (see Ghitany et al. (2008) for more details).

III. Some statistical properties

This section deal with some basic statistical properties of the proposed model like, its order relationship, moments, its conditional moments, its generating functions like moment and characteristics function, in measure of dispersion; mean deviation from mean and also from median, quantile function, distribution function of its order statistics and Shannon entropy. All the properties are given by one by one in subsequent subsections.

A. Some order relationship

Order relationship provide a thought to analyse the key features the distribution, density and hazard rate function of the RV. Before going to obtain its expressions, we recall its definitions as

A RV W is called to be smaller than another RV V in the following order term

- Stochastically ($W \leq_{ST} V$): if $F(w) \geq F(v) \quad \forall w$;
- In hazard rate ($W \leq_{HR} V$): if $h(w) \leq h(v) \quad \forall w$;
- In mean residual life ($W \leq_{MLR} V$): if $Mr(w) \leq Mr(v)$ $\forall w$, where $Mr(\cdot)$ is mean residual life;
- In likelihood ratio ($W \leq_{LR} V$): if $f(w)/f(v)$ decreasing in w.

Shaked and Shanthikumar (1994) show another relation between these is $W \leq_{ST} V \implies W \leq_{HR} V \implies W \leq_{LR} V$ and $W \leq_{HR} V \implies W \leq_{MLR} V$; see also, Ghitany et al. (2008), Gupta et al. (1998) etc. for order relationship.

Also, Ghitany et al. (2008) prove that Lindley distribution holds likelihood ratio ordering for $\theta_1 > \theta_2$ i.e. if two RVs X_1 and X_2 with PDF $g_1(x), g_2(x)$ and CDF $G_1(x), G_2(x)$ respectively and follow Lindley model having parameters value θ_1 and θ_2 and satisfied a condition $\theta_1 > \theta_2$ then, $g_1(x)/g_2(x)$ is decreasing and $G_1(x) \geq G_2(x)\ \forall x$. Then, $G_2(x)+1 \leq G_1(y)+1$ and this implies that

$$\frac{g_1(x)(G_2(x)+1)}{g_2(x)(G_2(x)+1)} = \frac{f_1(x)}{f_2(x)} < 1$$

where $f_1(x)$ and $f_2(x)$ are density functions of the RVs X_1 and X_2 of the proposed distribution. This result shows that for $\theta_1 > \theta_2$ the proposed distribution holds likelihood ratio ordering $X_1 \leq_{lr} X_2$ and hence holds all the above relationship like its baseline Lindley distribution. Now, using the above results, it can say that the proposed distribution holds stochastic order relationship also and it can verified by the Figure 1.

B. Moments of the distribution

Since moments of a distribution provides descriptive statistics and also it is useful in parameter estimation. This section discuss the expression for the moments of the proposed one with the help of a lemma.

Lemma III.1.

$$T_1(\theta, r, \delta) = \int_0^\infty \frac{x^r(1+x)e^{-\delta x}}{\left(2 - e^{-\theta x}\left(1+\frac{\theta x}{1+\theta}\right)\right)} dx,$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}(-1)^{k+l}\binom{k}{l}\binom{l}{m}\binom{m+1}{n} \times \frac{\theta^m}{(1+\theta)^l}\frac{(n+r)!}{(\delta+\theta l)^{n+r+1}}. \quad (6)$$

Proof. Using the expansion of $e^u = \sum_{l=0}^{\infty} u^l/l!$, one have

$$T_1(\theta, r, \delta) = \int_0^\infty x^r(1+x)e^{-\delta x}\sum_{k=0}^{\infty}(-1)^k\left[1 - e^{-\theta x}\left(1+\frac{\theta x}{1+\theta}\right)\right]^k dx$$

now, by the result of expansion of series, $(1-v)^b = \sum_{l=0}^{\infty}(-1)^l\binom{b}{l}v^l$, when b be a real number and $(1-v)^b = \sum_{l=0}^{b}(-1)^l\binom{b}{l}v^l$, when b be an integer number then simplifying, we have,

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}(-1)^{K+l}\binom{k}{l}\binom{l}{m}\binom{m+1}{n}\frac{\theta^m}{(1+\theta)^l}\frac{(n+r)!}{(\delta+\theta l)^{n+r+1}}.$$

(See Graham et al. (1989) for detail expression of series). □

According to the Lemma III.1, one have the r^{th} moment as,

$$E(X^r) = KT_1(\theta, r, \theta) \quad (7)$$

where $K = \dfrac{\theta^2}{\log 2(\theta+1)}$. Hence, the first moment i.e. arithmetic mean is, $E(X) = KT_1(\theta, 1, \theta)$. In the same way, other moments can also be obtained.

C. Conditional Moments of the distribution

The conditional moments can be used to find the moments under conditional probability, which is also helpful to find out the mean deviations. Again, a lemma is used to find the expression of it.

Lemma III.2.

$$T_2(\theta, r, \delta, t) = \int_t^\infty \frac{x^r(1+x)e^{-\delta x}}{\left(2 - e^{-\theta x}\dfrac{1+\theta+\theta x}{1+\theta}\right)} dx$$
$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}\sum_{p=0}^{n+r}(-1)^{k+l}\binom{k}{l}\binom{l}{m}\binom{m+1}{n}$$
$$\times \frac{\theta^m}{(1+\theta)^l}\frac{(n+r)!}{p!(\delta+\theta l)^{n+r+1}}e^{-(\delta+\theta l)t}\,[(\delta+\theta l)\,t]^p.$$

Proof.

$$T_2(\theta, r, \delta, t) = \int_t^\infty \frac{x^r(1+x)e^{-\delta x}}{\left(2 - e^{-\theta x}\dfrac{1+\theta+\theta x}{1+\theta}\right)} dx$$

A similar procedure have been taken as in Lemma III.1, the expression can be written as:

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}(-1)^{k+l}\binom{k}{l}\binom{l}{m}\binom{m+1}{n}$$
$$\times \frac{\theta^m}{(1+\theta)^l}\int_t^\infty x^{r+n}e^{-(\delta+\theta l)x}dx. \tag{8}$$

Here, we use complementary incomplete gamma function $\Gamma(b, y) = \int_y^\infty t^{b-1}e^{-t}dt$ which can be rewritten as $(a-1)!e^{-x}\sum_{l=0}^{a-1} x^l/l!$. By using this function in the equation (8) and after application of it, the above equation simplified as

$$T_2(\theta, r, \delta, t) = \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}\sum_{p=0}^{n+r}(-1)^{k+l}\binom{k}{l}\binom{l}{m}\binom{m+1}{n}$$
$$\times \frac{\theta^m}{(1+\theta)^l}\frac{(n+r)!}{p!(\delta+\theta l)^{n+r+1}}e^{-(\delta+\theta l)t}\,[(\delta+\theta l)\,t]^p.$$

□

Using the Lemma III.2, the r^{th} conditional moments could be easily find out as,

$$E(X^r|X > x) = \frac{K}{(1-F(x))}T_2(\theta, r, \theta, x). \tag{9}$$

D. Generating functions and characteristics function

The moment generating function (MGF) of RV X of proposed distribution is given as follows:

$$M_X(t) = KT_1(\theta, 0, \theta - t) \quad \text{for} \quad t < \theta.$$

Similarly the the cumulant generating function (CGF) of X is,

$$CG_X(t) = \log K + \log T_1(\theta, 0, \theta - t).$$

And the characteristic function (CHF) of X is,

$$\phi_X(t) = KT_1(\theta, 0, \theta - it).$$

where $i = \sqrt{-1}$ denotes imaginary value.

E. Mean deviation about its mean and median value

In descriptive statistics, measure of dispersion has as valuable as measure of central tendency. Mean deviation about central value i.e. mean and median are one of the measure of dispersion. The Mean deviation from mean (MDM) (ν) is defined as $L_1 = \int_0^\infty (x-\nu)f(x)dx$ and Mean deviation from median (MDMD) (M_d) $L_2 = \int_0^\infty (x - M_d)f(x)dx$. So by using method of integration by part, the MDM is,

$$L_1 = \int_0^\nu (\nu - x)f(x)dx + \int_\nu^\infty (x-\nu)f(x)dx.$$

Thus, since $\int_0^\nu f(x)dx = F(\nu)$

$$L_1 = 2\nu F(\nu) - 2\nu + 2\int_\nu^\infty xf(x)dx.$$

Now, considering the result of Lemma III.2,

$$\int_\nu^\infty f(x)dx = KT_2(\theta, 1, \theta, \nu)$$

and hence,

$$L_1 = 2\nu F(\nu) - 2\nu + KT_2(\theta, 1, \theta, \nu).$$

In the same fashion, the MDMD can be obtained as,

$$L_2 = \int_0^{M_d}(M_d - x)f(x)dx + \int_{M_d}^\infty (x - M_d)f(x)dx$$

the rest steps are similar to MDM, we have

$$L_2 = -\nu + 2\int_{M_d}^\infty xf(x)dx.$$

Now, using Lemma III.2,

$$\int_{M_d}^\infty xf(x)dx = KT_2(\theta, 1, \theta, M_d).$$

Hence, the MDMD is

$$L_2 = -\nu + KT_2(\theta, 1, \theta, M_d).$$

F. Quantile function

The q^{th} quantile $P(q)$ can be obtained by the equation $F(P(q)) = q$. So that, using the equation (3),

$$e^{-\theta P(q)}\frac{1+\theta+\theta P(q)}{1+\theta} = q \tag{10}$$

for $0 < q < 1$, we put $U(q) = -1 - \theta - \theta P(q)$ in equation (10) and put $T(q) = 2 - 2^q$, we get $U(q)e^{U(q)} = -(1+\theta)e^{-(1+\theta)}T(q)$ then solution for $U(q)$ is,

$$U(q) = W\left[-(1+\theta)e^{-(1+\theta)}T(q)\right] \tag{11}$$

here $W(\cdot)$ is Lambert W function, for more detail see Corless et al. (1996). Hence, from equation (11), quantile function is,

$$P(q) = -1 - \frac{1}{\theta} - \frac{1}{\theta}W\left[-(1+\theta)e^{-(1+\theta)}T(q)\right].$$

One can easily obtain median of the proposed model just by putting $q = 1/2$.

G. Order statistics of the distribution

Let $x_1, x_2, \ldots, x_m$ be m random sample from the proposed LoTL and their corresponding order statistics is, $x_{1:m} < x_{2:m} < \cdots < x_{m:m}$. Let $F(x)$ and $f(x)$ be the population CDF and PDF respectively, then for $p = 1, 2, \ldots, m$ the PDF $f_p(x)$ of p^{th} order statistics $X_{p:m}$ is,

$$f_p(x) = \frac{m!}{(p-1)!(m-p)!} F^{p-1}(x)[1-F(x)]^{m-p} f(x)$$
$$= \frac{m!}{(p-1)!(m-p)!} \sum_{i=0}^{m-p} (-1)^i \binom{m-p}{i} F^{p+i-1}(x) f(x). \tag{12}$$

Now by using equations (3) and (4) in (12) we have,

$$f_p(x) = \frac{m!}{(p-1)!(m-r)!} \frac{\theta^2}{(1+\theta)\log 2} \sum_{i=0}^{m-p} \frac{(-1)^i (1+x) e^{-\theta x}}{\left(2 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}\right)}$$
$$\times \binom{m-p}{i} \left[\frac{\log\left[2 - e^{-\theta x}\left(1 + \frac{\theta x}{1+\theta}\right)\right]}{\log 2} \right]^{p+i-1}.$$

And corresponding CDF $F_p(x)$ is,

$$F_p(x) = \sum_{l=p}^{m} \binom{m}{l} F^l(x)[1-F(x)]^{m-l}$$
$$= \sum_{l=p}^{m} \sum_{j=0}^{m-i} \binom{m}{l} \binom{m-l}{j} (-1)^j F^{l+j}(x). \tag{13}$$

Using equation (3) in equation (13) we have,

$$F_p(x) = \sum_{l=p}^{m} \sum_{j=0}^{m-l} \binom{m}{l} \binom{m-l}{j} \left[\frac{\log\left[2 - e^{-\theta x}\left(1 + \frac{\theta x}{1+\theta}\right)\right]}{\log 2} \right]^{l+j}.$$

H. Entropy for the distribution

Actually, entropy measures the amount of information contained in RV X on average. A famous entropy is Shannon entropy (proposed by Shannon (1951)), and defined as, $E[-\log f(x)]$. For the proposed model

$$-\log f(x) = -\log\left(\frac{\theta^2}{(1+\theta)\log 2}\right) - \log(1+x) + \theta x$$
$$-\log\left[2 - e^{-\theta x}\left(1 + \frac{\theta x}{1+\theta}\right)\right]$$

and hence,

$$E[-\log f(x)] = (1 - \log K) + \theta K T_1(\theta, 1, \theta) - K \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} T_1(\theta, i, \theta).$$

where $T_1(\cdot,\cdot,\cdot)$ has been define in Lemma III.1.

IV. Estimation procedure for the distribution

After studying various properties of model, it is important to estimate the unknown population parameter involve in model. One of the famous and used method is maximum likelihood estimation (MLE) which gives the value of parameter for which the likelihood function be maximum. Since maximizing likelihood is same as maximizing its logarithmic function. Thus, the logarithm likelihood function of the proposed distribution is,

$$\log L = n \log K - \theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log(1+x_i)$$
$$+ \sum_{i=1}^{n} \log\left(2 - \frac{1+\theta+\theta x_i}{1+\theta} e^{-\theta x_i}\right).$$

Now, differentiate the above one with respect to the parameter θ one can easily get,

$$\frac{\partial \log L}{\partial \theta} = \frac{n(\theta+2)}{\theta(\theta+1)} - \sum_{i=1}^{n} x_i - \frac{\theta e^{-\theta x_i}(1 + x_i(1+x_i)(1+\theta))}{(1+\theta)\left(2(1+\theta) - e^{-\theta x_i}(1+\theta+\theta x_i)\right)} \tag{14}$$

Now, equating the equation (14) to zero, we have a non-linear equation and its solution provide estimate $\hat{\theta}$ of the parameter θ. This also need some computational method to solve it because the likelihood equation is not in closed form. Here, we suggest to the use of Newton type method. This can be done by using R Core Team (2020) software. In the large samples, the confidence intervals can be calculated by using the Fisher information matrix $I^{-1}(\hat{\theta})$ by which one can get asymptotic variance for the estimated parameter. And, two-sided $100(1-\xi)\%$ confidence interval of θ is obtain as $\hat{\theta} \pm Z_{\xi/2}\sqrt{V(\hat{\theta})}$, where $Z_{\xi/2}$ stands for the upper $\xi/2\%$ points of standard normal distribution.

The estimated Fisher Information matrix is,

$$I(\hat{\theta}) = \left[\frac{-\partial^2 \log L}{\partial \theta^2}\right]_{\hat{\theta}}$$

where, $\frac{-\partial^2 \log L}{\partial \theta^2}$ is second derivative of logarithmic of likelihood function, obtained from equation (14).

V. Real data application

In this paper, two different real datasets in different fields to validate the applicability of the proposed one. Here, eight other lifetime models in which the only one has one parameter i.e. baseline Lindley distribution, otherwise, all other seven have two parameters have been considered for model comparison. The considered distributions are Chen model (Chen (2000)), Lindley, generalized Lindley (GL) (Nadarajah et al. (2011)), Weibull, gamma, GDUS exponential (GDUSE) (Maurya et al. (2017b)), gamma Lindley (GaL) (Zeghdoudi and Nedjar (2016)) and Power Lindley (PL) (Ghitany et al. (2013)). Among the compared models, two are famous bathtub models i.e. Chen and GDUSE models, three models are based on Lindley as a baseline, and two other flexible lifetime models as Weibull and gamma distributions.

A. Data Analysis

Data Set 1: Sand sample data.
These datasets contain 16 samples of maximum likeness estimates from the Danish west coast and reported by Barndorff-Nielsen (1977).

Data Set 2: Vinyl chloride data.
This data set is reported by Bhaumik et al. (2009). It consists 34 sample of vinyl chloride from clean up-gradient monitoring wells in mg/l.

Here, to know the type of the hazard rate of considered real datasets, we draw a scaled TTT plot in Figure 2 (See Aarset (1987) for more detailed about scaled TTT plot).

Here, firstly p value is calculated to validate that the model fit to dataset or not. After that, two model section criterion namely; AIC and BIC have been used. Also, we have used KS statistic and log likelihood value as a model selection criterion. Along with maximum likelihood estimates of the parameters all the above values are shown in Table I.

From this, we observed that for both datasets, all the considered models fitted at a 5% level of significance. And for the first dataset, the proposed LoTL model has minimum KS statistic and the log-likelihood value is almost same for seven models. But, the AIC and BIC values are least for the proposed LoTL model. For the second dataset, the KS statistics are minimum for the GaL distribution and the value of -LogL is almost same for five distributions. The AIC and BIC values are least for our proposed LoTL model.
To check the uniqueness properties of MLE, we have plotted log-likelihood value with a variation of a parameter value in Figure 3. This figure represents that the proposed model has a unique MLE. Also, we have considered non-parametric tools like the empirical CDF (ECDF) plots (see Figure 2) for all the considered models, kernel density (KD) plots, relative histogram plots along with fitted density plots of the proposed model have been presented in Figure 3. These figures also support our findings.

TABLE I
MLE, LOG LIKELIHOOD, KS STATISTICS WITH P-VALUE, AND AIC, BIC VALUES FOR THE DATASETS.

Dataset 1							
	MLE			KS			
Distribution	α	θ	-LogL	Statistic	p-value	AIC	BIC
Chen	0.408	0.038	59.949	0.169	0.692	123.898	125.443
Lindley	-	0.106	57.916	0.130	0.919	117.831	118.604
Proposed	-	0.122	57.905	0.121	0.951	117.810	118.583
GL	1.404	0.144	57.566	0.143	0.855	119.132	120.677
Weibull	1.482	17.383	58.073	0.147	0.832	120.145	121.690
Gamma	2.265	6.871	57.456	0.140	0.873	118.912	120.457
GDUSE	2.224	0.123	57.659	0.143	0.852	119.317	120.862
GaL	10.000	0.128	57.564	0.127	0.931	119.128	120.673
PL	1.087	0.095	57.777	0.141	0.864	119.554	121.099
Dataset 2							
Chen	0.506	0.303	57.975	0.115	0.755	119.950	123.002
Lindley	-	0.824	56.304	0.133	0.588	114.607	116.134
Proposed	-	0.725	55.658	0.106	0.843	113.315	114.841
GL	0.865	0.762	56.111	0.116	0.747	116.222	119.275
Weibull	1.010	1.888	55.450	0.092	0.937	114.899	117.952
Gamma	1.063	1.769	55.413	0.097	0.904	114.826	117.879
GDUSE	0.866	0.613	56.049	0.115	0.761	116.097	119.150
GaL	0.347	0.532	55.453	0.089	0.951	114.905	117.958
PL	0.883	0.914	55.760	0.094	0.923	115.520	118.573

VI. CONCLUSION

In this research paper, we propose an one parameter lifetime distribution that have capability of increasing and bathtub type hazard rates. We have derived some basic statistical properties like its moments, conditional moments, generating functions like; moment, cumulant and characteristic function, in measure of dispersion; mean deviation about mean and median value are calculated. The quantile function, Shannon entropy, the CDF and PDF of order statistic are derived. Along with this, method of obtaining the MLE, asymptotic confidence interval and observed Fisher information have also been discussed in detail. Two real datasets and eight other competitive distributions namely Lindley, Chen, GDUSE, generalised Lindley,

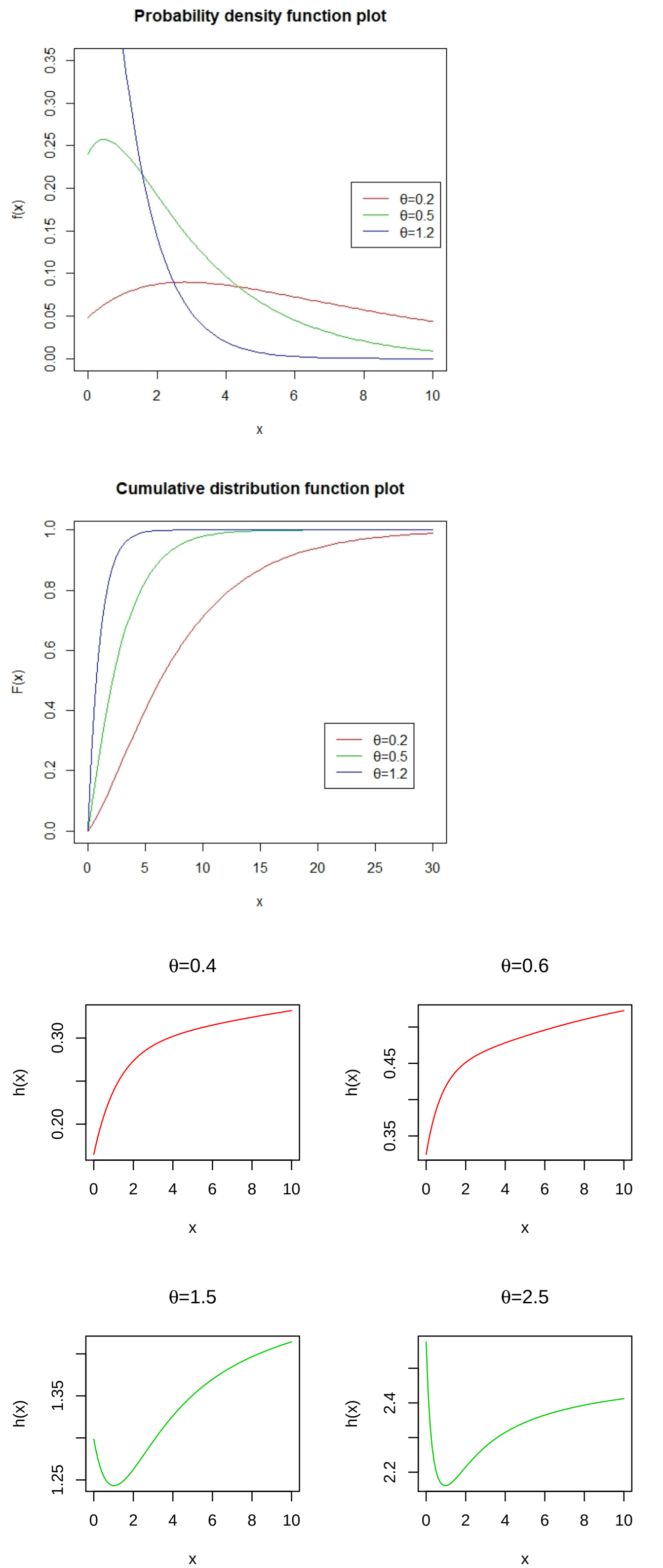

Fig. 1. Probability density and cumulative distribution function and hazard rate plot

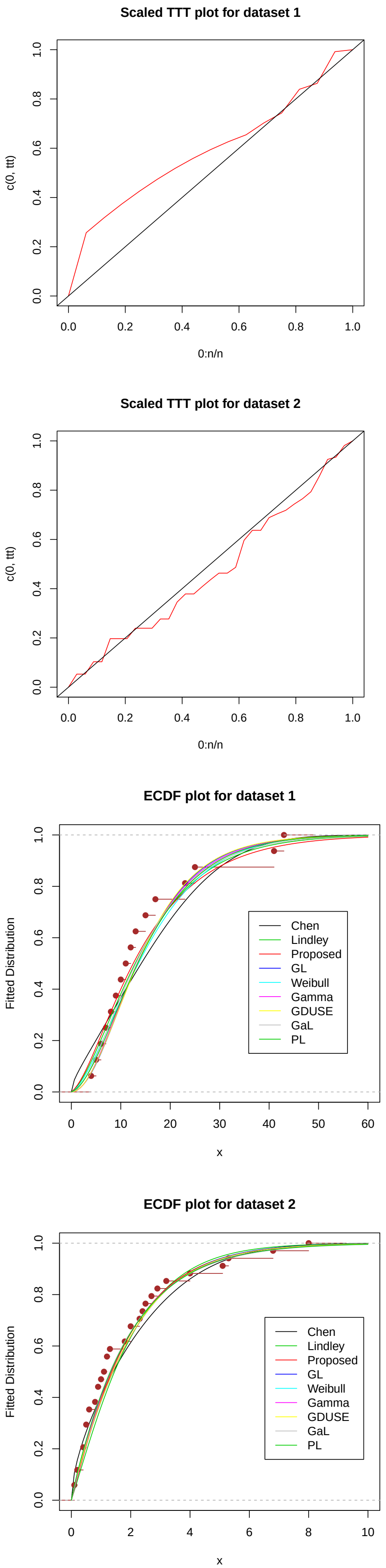

Fig. 2. Scaled TTT and ECDF plots for real datasets

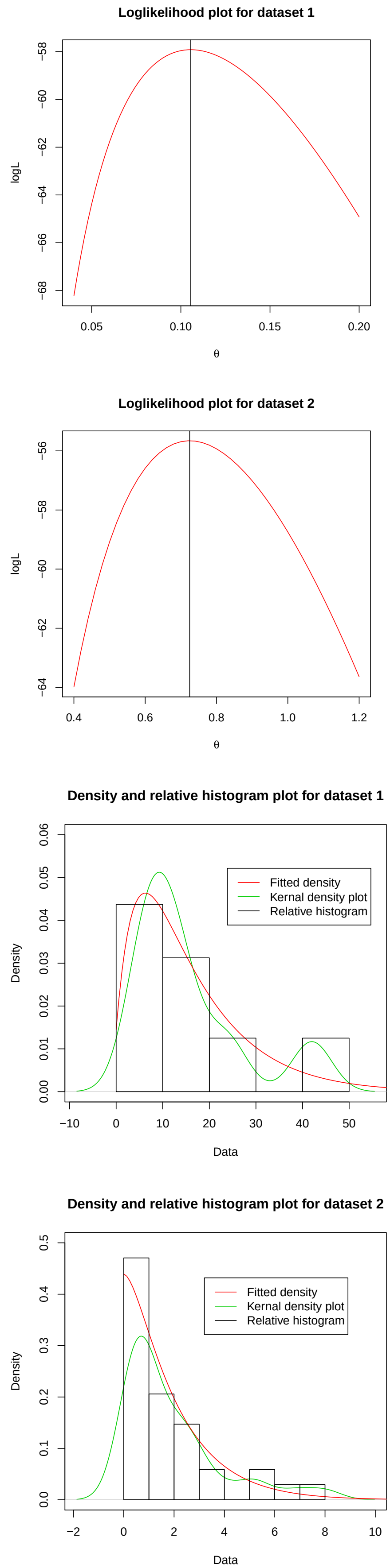

Fig. 3. Log likelihood, kernel density with fitted density and histogram plots

gamma and Weibull, Gamma Lindley and power Lindley have been considered for model validation. And it have shown via numerical measures as well as graphically that the proposed one fit to the considered real datasets very well than the other distributions. Hence, it is easily to conclude that the proposed distribution is flexible and fits variety of datasets and in this way be one may consider as a suitable model for lifetime data.

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