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# On reduction and relation type of an ideal

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***Abstract*—For an ideal $I$ of a Noetherian local ring $(R, m)$ we study properties of reduction of $I$. In particular, we investigate behaviour of the reduction of $I$ in term of relation type and defining equation of the Rees algebra. We also prove a variant of Northcott and Rees result.**

***Index Terms*—Analytic spread, Fiber cone, Rees algebras Reduction number, Relation type.**

## I. Introduction

For the ring $R$ and an ideal $I \subset R$ one can form the Rees algebra $R[It]$. The Rees algebra of an ideal is the commutative algebra analogue of the blow-up in algebraic geometry. Note that the projective scheme $Proj(R[It])$ defined by the Rees algebra $R[It]$ is the blowing-up of $Spec(R)$ along the variety $V(I)$.

A fundamental tool for examining properties of the Rees algebra of an ideal is the reduction of an ideal. The concept of reduction of ideals and integral closure were introduced by Rees [9]. There is a close connection between reduction of an ideal and module finite Rees algebra, where the ring is a Noetherian. This connection is very important for proving many results of Rees algebras. In Rees algebras, the Noether normalization lemma is key for finding dimension of the fiber cone and minimal reduction of an ideal.

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In the last few decades, the theory of reduction of ideals have been investigated by many researchers. However, the relation type of ideals has not been studied much. The relation type of ideals is useful to determine equation of the Rees algebra and reduction number. The equation of the Rees algebra is referred as generators of $ker(\phi) =: Q$, where $\phi : R[X_1, \ldots, X_r] \to R[It]$ and $I = (x_1, \ldots, x_r)$. The relation type of $I$, $rt(I)$ is the least integer $N \geq 1$ such that $Q = Q(N)$, where $Q(N)$ is the ideal generated by homogeneous equations of $R[It]$ of degree at most $N$. If $R$ is a Cohen-Macaulay ring, then equation of the Rees algebra is generated by Koszul relations [3]. In general, this result is not true, even though ring is not a Cohen-Macaulay and ideal is not a prime ideal. Later, Huneke [6] asked that if ring is a complete equidimensional Noetherian local ring, then there is a uniform bound for the relation type of ideals. Authors [1] showed that answer is affirmative. They constructed examples of parameter ideals of $R$ with unbounded relation type in non-Cohen-Macaulay locus of $R$ having dimension two or more. This result shows an idea of the complexity of the structure of the equation of the Rees algebra. Again, Hukaba [4] gave relation between reduction number and relation type and proved that the reduction of an ideal induced an equation of the Rees algebra of maximum degree. In [5] Heinzer and Kim proved that the equation of the fiber cone of an ideal is generated by a single equation of degree. In [8] authors described the equation of the Rees algebra of an equimultiple $I$ of deviation and proved that there is a unique equation of maximum degree in a minimal generating set of defining equation of the Rees algebra.

In this paper, we prove a variant of the result (Theorem 8.6.6, [10])) and study properties of reduction of ideals. We also describe defining equation of Rees algebra, when an ideal $I$ is of linear type.

## II. Preliminaries

In this section, we review some definitions, general facts about the Rees algebra, linear type of an ideal and the fiber cone. For more details the theory of reductions of ideals and linear type ([10]) are given. Let $R$ be a commutative ring with identity and $I$ be an ideal of $R$. The Rees algebra of $I$ is defined as

$$R[It] = \left\{ \sum_{i=0}^{n} a_i t^i \mid n \in \mathbb{N}, a_i \in I^i, a_0 \in R \right\} = \bigoplus_{n \geq 0} I^n t^n,$$

where $t$ is a variable over $R$.

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*Definition 2.1:* Let $J \subseteq I$ be ideals of a ring $R$. Then $J$ is said to a reduction of $I$ if $J\,I^n = I^{n+1}$ for some $n \geq 0$. The reduction number of $I$ with respect to $J$ is defined as

$$r_J(I) = Min\left\{n \geq 0 \mid J\,I^n = I^{n+1}\right\}$$

*Definition 2.2:* An ideal $J$ of $I$ is called a minimal reduction of $I$ if no ideal strictly contained in $J$ is a reduction of $I$.

*Remark 2.3:*

1) Every ideal is itself a reduction.
2) If $R$ is a Noetherian ring, then $R[It]$ is a module-finite $R[Jt]$ if and only if $J$ is a reduction of $I$ and $r_J(I)$ is the largest degree of an element in a homogeneous minimal generating set of the ring $R[It]$ over the ring $R[Jt]$.
3) If $n = r_J(I)$, then $J^m I^n = I^{m+n}$ for all $m \geq 0$.
4) In a Noetherian local ring,minimal reductions exist and it is unique minimal reduction of an ideal.

Suppose that $(R, m)$ is a Noetherian local ring. The fibre cone is defined as

$$F_I(R) = \frac{R[It]}{mR[It]} \simeq \bigoplus_{n\geq 0} \frac{I^n}{mI^n}$$

The Krull dimension of $F_I(R)$ is said to be analytic spread of an ideal $I$. It is denoted by $l(I) = l$.

*Definition 2.4:* Let $f : S(I) \longrightarrow R(I)$ be a canonical morphism from the Symmetric algebra $S(I)$ to the Rees algebra $R(I)$. An ideal $I$ is said to be a linear type if $f$ is an isomorphism i.e $S(I) \simeq R(I)$.

*Proposition 2.5:* If $(R, m)$ is a Noetherian local ring having infinite residue field $R/m =: k$ and $J \subset I$ is a minimal reduction of $I$, then

1) $J \cap mI = mJ$.
2) $\mu(I) = \mu(J) + \mu(I/J)$.

*Proof:*

1) Let $L = J \cap mI$. Consider the map $\phi : \frac{J}{mJ} \to \frac{J}{L}$. Then the map $\phi$ is onto and $l_R(J/L)$ is finite, where $l_R(.)$ denotes the length of an $R$-module. So that $\frac{J}{L} \simeq (R/m)^s$ for some $s > 0$. Therefore $J = (x_1, \ldots, x_s) + L$, where $x_i \in J$. We have to show that $K = (x_1, \ldots, x_s) \subseteq J$ is a reduction of $I$. Since $J$ is a reduction of $I$, $I^{n+1} = JI^n \subseteq (K + mI)I^n \subseteq KI^n + mI^{n+1} \subseteq I^{n+1}$. Therefore $K$ is a reduction of $I$. Moreover by minimality of $J$ we must $J = K$ and $L = mJ$.
2) By isomorphism theorem and (1), we have $\frac{J+mI}{mI} \simeq \frac{J}{J\cap mI} \simeq \frac{J}{mJ}$. Therefore, $\frac{J+mI}{mI} \simeq \frac{J}{mJ}$. Consider short exact sequence

$$0 \longrightarrow \frac{J}{mJ} \longrightarrow \frac{I}{mI} \to \frac{I}{mI+J} \longrightarrow 0.$$

Note that $\frac{I}{mI+J} \simeq \frac{I/J}{m(I/J)}$. Hence$\mu(I) = dim_k(\frac{I}{mI}) = dim_k(\frac{J}{mJ}) + dim_k(\frac{I/J}{m(I/J)} = \mu(J) + \mu(I/J)$.

■

*Corollary 2.6:* Suppose $(R, m)$ is a Noetherian local ring and $J$ is reduction of $I$. Then $J \nsubseteq mI$.

*Proof:* We may assume that $K \subseteq J$ is a minimal reduction of $I$. Then we have to show that $K \nsubseteq mI$. Suppose the contrary that $K \subseteq mI$. By Proposition 2.5, $K \cap mI = mK$ and we have $K = mK$. Hence $K = 0$ ( Nakayama's Lemma), which is a contradiction. Therefore $K \nsubseteq mI$ and so $J \nsubseteq mI$.

■

## III. Reductions of ideals

In this section, we give behaviour of reduction of ideals of a linear type and a variant result of North-Cott. Now, we prove the following Theorem 3.1 by using the notions of Remark 2.3.

*Theorem 3.1:* Let $(R, m)$ be a Noetherian local ring with infinite residue field $R/m$ and $J \subseteq I$ be a reduction of $I$. Then

$$l = Min\{\mu(J) | J\ is\ a\ reduction\ of\ I\},$$

where $\mu(J)$ is the minimal number of generators of $J$.

*Proof:* Let $J$ be a reduction of $I$. Then $R[It]$ is module-finite over $R[Jt]$ (Remark 2.3). Consider the inclusion map $\phi : R[Jt] \hookrightarrow R[It]$ and its induced map $\bar{\phi} : \frac{R[Jt]}{mR[It]\cap R[Jt]} \longrightarrow \frac{R[It]}{mR[It]}$ such that$\bar{\phi}(\sum a_i t^i + mR[It] \cap R[Jt]) = \phi(\sum a_i t^i) + mR[It] = \sum a_i t^i + mR[It]$, where $a_i \in J^i$. Claim: $\bar{\phi}$ is a well defined map. Suppose $\sum a_i t^i + mR[It] \cap R[Jt] = \sum b_i t^i + mR[It] \cap R[Jt]$. Then $\sum a_i t^i - \sum b_i t^i \in mR[It] \cap R[Jt]$. Therefore $\sum a_i t^i - \sum b_i t^i \in R[Jt]$ and $\sum a_i t^i - \sum b_i t^i \in mR[It]$. Hence $\sum a_i t^i + mR[It] = \sum b_i t^i + mR[It]$. It follows that $\bar{\phi}(\sum a_i t^i + mR[It] \cap R[Jt]) = \bar{\phi}(\sum b_i t^i + mR[It] \cap R[Jt])$. So $\bar{\phi}$ is a well defined map. Since diagram

$$\begin{array}{ccc} R[Jt] & \xrightarrow{\phi} & R[It] \\ \downarrow{\psi_1} & & \downarrow{\psi_2} \\ \frac{R[Jt]}{mR[It]\cap R[Jt]} & \xrightarrow{\bar{\phi}} & \frac{R[It]}{mR[It]} \end{array}$$

commutes i.e. $\psi_1 o \phi = \bar{\phi} o \psi_2$, $F_I(R)$ is a module-finite over $\frac{R[Jt]}{mR[It]\cap R[Jt]}$ and $l(I) = dim(F_I(R)) = dim\left(\frac{R[Jt]}{mR[It]\cap R[Jt]}\right)$. But $dim\left(\frac{R[Jt]}{mR[It]\cap R[Jt]}\right) \leq \mu(J)$. Indeed, if $\{\frac{a_1+mI}{mI}, \ldots, \frac{a_r+mI}{mI}\}$ is the minimal generating set of $\frac{J+mI}{mI}$, then we can find map $\frac{R}{m}[X_1, \ldots, X_r] \xrightarrow{T} \frac{R}{m} \oplus \frac{J+mI}{mI} \oplus ... \cong \left(\frac{R[Jt]}{mR[It]\cap R[Jt]}\right)$ given by $X_i \longrightarrow \frac{a_i+mI}{mI}$. Clearly the map $T$ is onto. Therefore, $dim\left(\frac{R[Jt]}{mR[It]\cap R[Jt]}\right) \leq dim\left(\frac{R}{m}[X_1, \ldots, X_r]\right) = r = dim_{R/m}\left(\frac{J+mI}{mI}\right) = \mu(J)$. Therefore $l(I) \leq \mu(J)$. We have to show that $J$ is the minimal reduction of $I$ generated by $l(I) = l$ elements. First claim: $J$ is a reduction generated by $l(I) = l$ elements. Let $\{x_1, \ldots, x_s\}$ be a minimal generating set of $I$. Then $\{\overline{x_1}, \ldots, \overline{x_s}\}$ is a basis of the vector space $I/mI$ over $R/m$, where $\overline{x_i} = x_i + mI$. Therefore, $F_I(R)$ is a finitely generated $R/m$-algebra generated by $\overline{x_1}, \ldots, \overline{x_s}$ elements. By the Noether's normalization lemma, there exist $\overline{x_1}, \ldots, \overline{x_s} \in F_I(R)$ which are algebraically independent over $R/m$ such that $F_I(R)$ is a module-finite over $R/m[\overline{x_1}, \ldots, \overline{x_s}]$ and $dim(R/m[\overline{x_1}, \ldots, \overline{x_s}]) = s = l$. Hence $\overline{x_1}, \ldots, \overline{x_l}$ generates $F_I(R)$. Therefore, $F_I(R)_{n+1} = (\overline{x_1}, \ldots, \overline{x_l})F_I(R)_n$. This implies that $\frac{I^{n+1}}{mI^{n+1}} = (\overline{x_1}, \ldots, \overline{x_l})\left(\frac{I^n}{mI^n}\right)$ for some $n \geq 0$. Hence

$$\frac{I^{n+1}}{mI^{n+1}} \subseteq \frac{(x_1, \ldots, x_l)I^n + mI^{n+1}}{mI^{n+1}}$$

and $I^{n+1} \subseteq (x_1, \ldots, x_l)I^n + mI^{n+1} \subseteq I^{n+1}$. So inequality holds throughout and we have, $JI^n = I^{n+1}$, where $(x_1, \ldots, x_l) = J$. Hence $J$ is a reduction of $I$ generated by $l(I) = l$ elements.

Second claim: There is no reduction generated by less than $l(I)$ elements. Suppose contrary by that $J$ is a reduction of $I$ generated by $r < l(I)$ elements. Then for any reduction $J$ of $I$ we have $l(I) \leq \mu(J)$, which is a contradiction. Hence $J$ is the minimal reduction of $I$ and $l(I) = \mu(J)$. ■

Following example shows that $l(I) = l(J)$ but we do not always have $J$ is a reduction of $I$.

*Example 3.2:* Let $R = k[[X_1, X_2]]$ be a ring over field $k$ and $J = (X_1^2, X_2) \subset I = (X_1, X_2)$ be ideals of $R$. Then $l(I) = l(J) = 2$ but $J$ is not a reduction of $I$.

*Proposition 3.3:* Suppose $(R, m)$ is a Noetherian local ring having infinite residue field $R/m$ and $I$ is of linear type ideal in $R$. Then $l(I) = \mu(I)$ and $I$ is itself a minimal reduction.

*Proof:* Let $I$ be of linear type. Then $R[It] \simeq S(I)$ and the analytic spread of an ideal

$$\begin{aligned} l(I) &= dim(R[It] \otimes_R R/m) \\ &= dim(S(I) \otimes_R R/m) \\ &= dim_{R/m}(I \otimes_R R/m) \\ &= dim_{R/m}(\frac{I}{mI}). \end{aligned}$$

Since $I/mI$ is a finite dimensional vector space over field $R/m$, $\mu(I) = dim_{R/m}(\frac{I}{mI}) = l(I)$. Therefore $\mu(I) = l(I)$. Assume that $J$ is a minimal reduction of $I$. By Theorem 3.1, $l(I) = \mu(J)$, $\mu(I) = \mu(J)$. So that $dim_{R/m}(I/mI) = dim_{R/m}(J/mJ)$ and by Proposition 2.5(1), $dim_{R/m}(J + mI/mI) = dim_{R/m}(J/J \cap mI) = dim_{R/m}(J/mJ) = dim_{R/m}(J/mJ)$. Therefore $dim_{R/m}(J + mI/mI) = dim_{R/m}(I/mI)$ and we deduce that $J + mI = I$. By Nakayama's lemma, $J = I$. ■

*Proposition 3.4:* Let $R$ be a Noetherian ring and $I$ be of linear type ideal in $R$. Then relation type of $I$ is one.

*Proof:* Let $I = (x_1, \ldots x_r)$ be an ideal and $X_1, \ldots, X_r$ be variables over $R$. Consider the ring homomorphism $\phi : R[X_1, \ldots, X_r] \to R[It]$ defined by $X_i \to x_i t$. Then $\phi$ is a onto $R/m$-algebra homomorphism and its kernel $Q$ is a homogeneous ideal. If $Q(1)$ is a homogeneous component of $Q$ with degree 1, then $Q(1) = \{a_1 X_1 + \cdots + a_r X_r \mid a_i \in R, a_1 x_1 + \cdots + a_r x_r = 0\}$. Since $Q(1) \subset Q$, we have commutative diagram

$$\begin{array}{ccc} \dfrac{R[X_1, \ldots, X_r]}{Q(1)} & \longrightarrow & \dfrac{R[X_1, \ldots, X_r]}{Q} \\ \big\downarrow & & \big\downarrow \\ S(I) & \longrightarrow & R[It] \end{array}$$

Since $I$ is of linear type, $S(I) \simeq R[It]$. Therefore $\frac{R[X_1, \ldots, X_r]}{Q(1)} \simeq \frac{R[X_1, \ldots, X_r]}{Q}$ and $Q(1) = Q$. So that $rt(I) = 1$. Note that $rt(I) = r_J(I) + 1$ for every reduction $J$ of $I$. By Proposition 3.3, $I$ is itself minimal reduction and $r_J(I) = 0$. Again we can see that $rt(I) = 0 + 1 = 1$. ■

*Corollary 3.5:* Suppose $R$ is a Noetherian ring and $I$ is an ideal of linear type. For all prime ideal $p \in Spec(R)$ containing $I$, $I_p$ is a minimal reduction itself.

*Proof:* Note that $(R_p, pR_p)$ is a Noetherian local ring having infinite residue field $R_p/pR_p$. Since $I$ is of linear type, $I_p$ is of linear type for all $p \in Spec(R)$. By Proposition 3.3, $I_p$ is itself a minimal reduction. ■

If $I$ is an ideal of linear type, then $\mu(I_p) = l(I_p)$ for each prime ideal $I \subset p$ (Corollary 3.5). In general, the converse is not true as follows:

*Example 3.6:* Let $R = \frac{k[[X_1, X_2]]}{(X_1^2 X_2, X_1 X_2^2)} = k[[x_1, x_2]]$ be a ring. Suppose that $I = (x_2)$ and $p = (x_1, x_2)$ are ideals of $R$. Then $l(I_p) = \mu(I_p) = 1$ and $S(I) \simeq R(I)$. Therefore $I$ is not of linear type.

*Lemma 3.7:* Suppose $(R, m)$ is a Noetherian local ring and $\frac{R}{m} := k$ is infinite residue field. Let $l$ and $s$ are fixed positive integers and define the set $Y = \{A \in M_{l \times s}(k) \mid A : k^s \longrightarrow k^l$ is onto $\}$. Then

1) the set $Y$ is a open subset of $k^{ls}$.
2) the set $Y$ is a non-empty if and only if $l \leq s$.

*Proof:*

1) Let $\{v_1, \ldots, v_s\}$ be fixed basis of $k^s$. Then we have a basis $\{x_1, \ldots, x_l\}$ of $k^l$ such that

$$x_i = \sum_{j=1}^{s} a_{ij} v_j, \forall\ i = 1, \ldots, l,\ where\ a_{ij} \in k.$$

Consider a map $A = \begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{sl} \end{bmatrix} : k^l \longrightarrow k^s.$

Since $A$ is *onto*, $rank(A) = l$. By definition of the rank, the basic open set of $k^{ls}$ is of the form

$$U = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{l1} & \cdots & a_{ls} \end{bmatrix} \mid \begin{vmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{l1} & \cdots & a_{ls} \end{vmatrix} \neq 0 \right\}.$$

Consequently, the number of basic open sets are $\binom{s}{l}$.

So that $Y$ is a open set and it is the union of all $\binom{s}{l}$ basic open sets.

2) We have to show that $Y$ is a non-empty open set. Consider the matrix

$$A = [a_{ij}]_{l\times s}$$

defined as follows

$$a_{ij} = \left\{ \begin{array}{ll} 1 & ; i = j \\ 0 & ; i \neq j \end{array} \right\}$$

Thus $A$ has block form $\begin{bmatrix} [I]_{l\times l} & [O]_{l\times s-l} \end{bmatrix}$, where $l \leq s$. Clearly

$$A : k^s \longrightarrow k^l$$

is onto. Therefore $A \in Y$ and $Y$ is a non empty set. Conversely, suppose $Y$ is a non empty set and $A \in Y$. Then $rank(A) = l$ and the $Rank$ and $Nulity$ Theorem, $l \leq s$

■

*Theorem 3.8:* If $(R, m)$ is a Noetherian local ring having infinite field $R/m := k$ and $J = (x_1, \ldots, x_s) \subset I$ are ideals of $R$, then the set

$$U = \left\{ \begin{bmatrix} \overline{u_{11}} & . & . & \overline{u_{1s}} \\ \overline{u_{21}} & . & . & \overline{u_{2s}} \\ . & . & . & . \\ . & . & . & . \\ \overline{u_{l1}} & . & . & \overline{u_{ls}} \end{bmatrix} \in A_{l\times s}(k) \;\middle|\; \begin{bmatrix} u_{11} & . & . & u_{1s} \\ u_{21} & . & . & u_{2s} \\ . & . & . & . \\ . & . & . & . \\ u_{l1} & . & . & u_{ls} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_s \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_l \end{bmatrix} \text{ and } (b_1, \ldots, b_l)I^n = I^{n+1} \right\}$$

is a non empty open subset of $k^{ls}$ if $l \leq s$, where $l$ and $s$ are fixed positive integers.

*Proof:* First we have to show that $U$ is a well defined set. Indeed, if

$$\begin{bmatrix} \overline{u_{11}} & . & . & \overline{u_{1s}} \\ \overline{u_{21}} & . & . & \overline{u_{2s}} \\ . & . & . & . \\ . & . & . & . \\ \overline{u_{l1}} & . & . & \overline{u_{ls}} \end{bmatrix} = \begin{bmatrix} \overline{v_{11}} & . & . & \overline{v_{1s}} \\ \overline{v_{21}} & . & . & \overline{v_{2s}} \\ . & . & . & . \\ . & . & . & . \\ \overline{v_{l1}} & . & . & \overline{v_{ls}} \end{bmatrix},$$

then for all $i$, $j$, $u_{ij} - v_{ij} \in m$ and

$$\begin{bmatrix} u_{11} & . & . & u_{1s} \\ u_{21} & . & . & u_{2s} \\ . & . & . & . \\ . & . & . & . \\ u_{l1} & . & . & u_{ls} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_s \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_l \end{bmatrix}$$

$$and \begin{bmatrix} v_{11} & . & . & v_{1s} \\ u_{21} & . & . & u_{2s} \\ . & . & . & . \\ . & . & . & . \\ v_{l1} & . & . & v_{ls} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_s \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_s \end{bmatrix}.$$

Therefore, $b_i = u_{i1}x_1 + u_{i2}x_2 + \cdots + u_{is}x_m$, $c_i = v_{i1}x_1 + v_{i2}x_2 + \cdots + v_{is}x_s$ and $b_i - c_i = (u_{i1} - v_{i1})x_1 + \cdots + (u_{is} - v_{is})x_s$. Note that $u_{ij} + m = v_{ij} + m$ and $b_i - c_i \in mI$. Then $(c_1, \ldots, c_l)I^n = I^{n+1}$ (Proposition3.2, [12]). Since $I = (x_1, \ldots, x_s)$, $I^n =< x_1^{r_1} \ldots . x_s^{r_s} \mid r_1 + r_2 + \cdots + r_s = n >$ and $I^{n+1} =< x_1^{l_1} \ldots . x_m^{l_s} \mid l_1 + l_2 + \cdots + l_s = n + 1 >$.

Assume that the minimal generating sets of $I^n$ and $I^{n+1}$ are fix and $b_1, \ldots, b_r \in I$. Consider the map $\left(\frac{I^n}{mI^n}\right)^l \xrightarrow{f} \frac{I^{n+1}}{mI^{n+1}}$ such that $f(\overline{y_1}, \ldots, \overline{y_l}) = \overline{b_1y_1 + b_2y_2 + \cdots + b_ly_l} = b_1y_1 + \cdots + b_ly_l + mI^{n+1}$. Note that the map $f$ is a well defined. Now $f(\overline{y_1}, \ldots, \overline{y_l}) = \overline{b_1y_1 + b_2y_2 + \cdots + b_ly_l} = \overline{b_1}\overline{y_1} + \cdots + \overline{b_l}\overline{y_l} = (\overline{u_{11}}\ \overline{x_1} + \cdots + \overline{u_{1s}}\ \overline{x_s})\overline{x_1^{r_1}x_2^{r_2} \ldots . x_s^{r_s}} + \cdots + (\overline{u_{l1}}\ \overline{x_1} + \cdots + \overline{u_{ls}}\ \overline{x_s})\overline{x_1^{l_1}x_2^{l_2} \ldots . x_s^{l_s}} = (\overline{u_{11}} + \overline{u_{21}} + \cdots + \overline{u_{l1}})\overline{x_1^{l_1+1}x_2^{l_2} \ldots . x_s^{l_s}} + \cdots + (\overline{u_{1s}} + \overline{u_{2s}} + \cdots + \overline{u_{ls}})\overline{x_1^{l_1}x_2^{l_2} \ldots . x_s^{l_s+1}}$, where $b_i = u_{i1}x_1 + \ldots, u_{is}x_s$. Therefore the coefficients of monomials $\overline{x_1^{l_1+1}x_2^{l_2} \ldots . x_s^{r_s}}$, $\overline{x_1^{l_1}x_2^{l_2+1} \ldots . x_s^{l_s}}, \ldots, \overline{x_1^{l_1}x_2^{l_2} \ldots . x_s^{r_s+1}}$ are linear polynomials.

It suffices to show that $f$ is onto if and only if $(b_1, \ldots, b_l)I^n = I^{n+1}$. Let $(b_1, \ldots, b_l)I^n = I^{n+1}$. Then we have to show that the map $f$ is onto. If $\overline{x} \in \frac{I^{n+1}}{mIn + 1}$, where $\overline{x} = x + mI^{n+1}$ and $x \in I^{n+1}$, then $x \in (b_1, \ldots, b_l)I^n$, for $I^{n+1} = (b_1, \ldots, b_l)I^n$. Therefore, $x = (b_1a_1 + \cdots + b_la_l)y = b_1a_1y + \cdots + b_la_ly$. Take $y_1 = a_1y, y_2 = a_2y, \ldots, y_l = a_ly$, where $a_i \in R$ and $y \in I^n$. It follows that if $\overline{x} = b_1y_1 + \cdots + b_ly_l + mI^{n+1}$ and $(\overline{y_1}, \ldots, \overline{y_l}) \in \left(\frac{I^n}{mI^n}\right)^l$, then $f(\overline{y_1}, \ldots, \overline{y_l}) = \overline{x}$. Therefore, the map $f$ is onto. Conversely, if $f$ is onto, then for any $\overline{x} \in I^{n+1}/mI^{n+1}$ there exists $(\overline{y_1}, \ldots, \overline{y_l}) \in \left(\frac{I^n}{mI^n}\right)^l$ such that $f(\overline{y_1}, \ldots, \overline{y_l}) = \overline{x}$. So that $\overline{x} = b_1y_1 + \cdots + b_ly_l + mI^{n+1}$ and $I^{n+1} \subseteq (b_1, \ldots, b_l)I^n + mI^{n+1}$. We go modulo $(b_1, \ldots, b_l)I^n$, $\frac{I^{n+1}}{(b_1, \ldots, b_l)\ I^n} \subseteq \frac{(b_1, \ldots, b_l)\ I^n + mI^{n+1}}{(b_1, \ldots, b_l)\ I^n} = m\left(\frac{I^{n+1}}{(b_1, \ldots, b_l)\ I^n}\right)$. By Nakayama lemma, $\frac{I^{n+1}}{(b_1, \ldots, b_l)\ I^n} = 0$. Therefore, $I^{n+1} = (b_1, \ldots, b_l)\ I^n$. In this case, we have $(b_1, \ldots, b_l)$ is a reduction of $I$. By assumption of $U$, we have

$$U = \left\{ \begin{bmatrix} \overline{u_{11}} & . & . & \overline{u_{1s}} \\ \overline{u_{21}} & . & . & \overline{u_{2s}} \\ . & . & . & . \\ . & . & . & . \\ \overline{u_{l1}} & . & . & \overline{u_{ls}} \end{bmatrix} \in A_{l\times s}(k) \;\middle|\; \begin{bmatrix} u_{11} & . & . & u_{1s} \\ u_{21} & . & . & u_{2s} \\ . & . & . & . \\ . & . & . & . \\ u_{l1} & . & . & u_{ls} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_s \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_l \end{bmatrix} \text{ and } f \text{ is onto} \right\}.$$

By Lemma 3.7, $U$ is a non empty open subset of $k^{ls}$ if $l \leq s$. ■

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