Volume 65, Issue 5, 2021
Journal of Scientific Research
of
The Banaras Hindu University

Bayesian Analysis of Nakagami Distribution under Generalized Type-II Progressive Hybrid Censoring Scheme

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Abstract: **The paper presents the two parameter Nakagami distribution as an efficient alternative lifetime model for generalized type-II progressive hybrid censored data. Such a censoring mechanism is most appropriate for recording failure times of robust items which have long lifetimes. Maximum likelihood and Bayes estimates are developed for the unknown parameters including the case of both parameters unknown. Posterior analysis is conducted under squared error and Linear exponential loss functions. A simulation study and two real data based empirical assessment of the theoretic findings of the paper are undertaken as illustrations.**

Index Terms: **Bayesian Parametric Estimation, Generalized Progressive Hybrid Censoring, Linear exponential loss function, Maximum Likelihood Estimate, Squared Error Loss.**

I. Introduction

Lifetime data represent waiting time to occurrence of an event of interest such as death, divorce, breakdown of a machine component, radioactive decay of unstable atoms, surging or plummeting of stock price and so on. Various social, biological, economic and industrial events fit to the diverse lifetime distributions with varying degrees. Best fit enhances predictive capability of the most closely fitted model. This has led to continuous pursuit of a better fitting model for individual events among researchers. Over the last six decades several new continuous distributions for modelling and analysis of lifetime data have been proposed. Extended versions of existing models such as beta modified, transmuted, exponentiated and generalized distributions represent concerted endeavour to address enhanced flexibility concerns with respect to skewness and kurtosis to specific applications and to the recorded data sets. Different methods of extending baseline distributions are available in literature. See for instance, Eugene et al. (2002), Jones (2004), Ghitany et al. (2007), Xhang and Xie (2007), Cordeiro and de Castro (2011), Nadarajah and Eljabri (2013), Sarhan and Akaloo (2013), Gomes et al. (2014) and Nadarajah et al. (2015). Distributions belonging to the skewed family have been found to be more flexible and versatile than the simple probability distributions by the applied data scientists.

It is common in life-testing and reliability studies to have incomplete information on failure times as some experimental units are reported as lost due to unknown causes or intentionally removed before termination of the experiment. Test units for which failure time information is not obtained are called censored units and the resulting data are called censored data. For highly reliable and durable products, the life-times are very long. Therefore, to optimize experimental time and cost, the test units are removed according to a well-defined mechanism called censoring scheme. The conventional Type-I, Type-II and hybrid (mixture of Type-I and Type-II) censoring schemes do not allow removal of test units during the conduct of life-test. For highly reliable products, *Progressive Type-II censoring* scheme permits removal of a pre-determined number of test units undergoing life test, at each observed failure. Thus, from a sample of n independent and identical units undergoing life test, the number of random removals R_i at the i^{th} failure, $i = 1,2,\ldots,m$ and the number of failures to be observed m are fixed in advance. R_1 units are randomly removed immediately at the point of first failure, $X_{1:m:n}$, from the $n-1$ surviving units. At the second failure, $X_{2:m:n}$, R_2 units are randomly removed from the $n-2-R_1$ surviving units. At the m^{th} failure, $X_{m:m:n}$ all remaining survivors $R_m = n-m-R_1-\cdots R_{m-1}$ are removed and the test terminates. When $R_1=\cdots=R_m=0$ and $m=n$, it is the complete sample while $R_1=\cdots=R_{m-1}=0$, and $R_m=n-m$ leads to the conventional Type-II censoring scheme or failure censoring which has a major disadvantage that if the last failure takes long time, then termination time of the experiment is also prolonged.

Kundu and Joarder (2006) proposed a *progressive Type II hybrid* censoring scheme in which the experiment on life-test is terminated at a $min\{X_{m:m:n}, T\}$, where $T \in (0,\infty)$ represents a pre-specified time point and $X_{i:m:n}$ denotes time of i^{th} failure.

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However, there is a possibility that very few failures may occur before time T, which leads to very few observed failures. Cho *et al*. (2015) proposed a *generalized Type-I progressive hybrid* censoring scheme which ensures a specified number of failures $k < m$, by defining termination time under this scheme as $max\{X_{k:m:n}, min\{X_{m:m:n}, T\}\}$, where T and k are pre-decided. *Generalized Type-II hybrid censoring* scheme proposed by Chandrashekhar *et al*. (2004) ensures observing sufficient failure counts while simultaneously capping the experimental time by proposing termination of a life-time experiment at $max\{min\{X_{r:n}, T_2\}, T_1\}$ for predetermined $T_1, T_2 \in (0, \infty)$ such that $T_1 < T_2$ and $1 \le r \le m$. Gòrny and Cramer (2016) included progressive removals aspect in it, to decide terminal point of the life test under *generalized Type-II progressive hybrid censoring* (GTPH) described as $max\{min\{X_{m:m+Rm:n}, T_2\}, T_1\}$. Using this censoring scheme, they determined density functions of the MLEs for both the location and the scale parameters of exponential distribution using spacing-based approach. Independently Lee *et al*. (2016) extended *Type-II progressive hybrid censoring* to the GTPH and considered exact as well as approximate inference for bias-adjusted parameters of exponential distribution. Seo and Kim (2017) proposed a robust Bayesian point estimation approach founded on a hierarchical structure for two parameter exponential distribution under GTPH. Koley and Kundu (2017) analysed *generalized progressively censored* data in the presence of competing risk. Kotb (2018) gave Bayesian prediction bounds for exponential type distribution under *generalized progressively hybrid censored* (GPH) scenario. Wang (2018) and Wang (2020) considered competing risk Weibull failure times model under GPH for complete and under partially observed failure causes respectively.

The present paper focuses on Bayesian parametric estimation under GTPH for two parameter Nakagami distribution (ND). Numerous empirical experiments have demonstrated applicability of ND in the scientific fields of wireless communications, wave propagation, intensity distribution due to rapid signal fading, modelling the echo from tissue and ultrasonic tissue characterization; see for instance, Shankar *et al.* (2001), Karagiannidis (2006), Tsui *et al.* (2010) among others. In this paper ND - a distribution popular in medical and wave theory sciences, is elaborately studied for possible use in reliability engineering and mechanical hardware modelling under GTPH and compared with some other established lifetime models to assert its suitability as a prospective lifetime model.

The rest of the paper is organised as follows: Section 2 details GTPH. ND model is introduced under section 3 along with its reliability characteristics. Maximum likelihood estimates (MLE) of the unknown parameters are derived under GTPH in section 4. Corresponding Bayesian parametric estimates under Squared Error (SELF) and Linear Exponential (LINEX) loss functions are presented under section 5. Section 6 details simulation study which is conducted to illustrate the theoretical inferences drawn. Two real data sets are explored to establish the findings of the present study in section 7. Section 8 concludes the achievements of the present paper.

II. Generalized Type-II Progressive Hybrid Censoring Scheme

The test experiment begins after fixing integer $m(\le n)$ and two threshold times T_1 and T_2 such that $0 < T_1 < T_2 < \infty$. The following three cases may arise under GTPH:

(i) The first m failures are observed with predetermined random withdrawals $R_i, i = 1, \dots, m$, if the m^{th} failure occurs before time T_1 such that further failures are continued to be observed but without random removals. This censoring procedure is represented as ; $R^* = (R_1, \dots, R_{m-1}, 0^{0*(R_m+1)})$, where $0^{0*(R_m+1)}$ denotes a vector of $R_m + 1$ zeros.

(ii) If the m^{th} failure time is located between times T_1 and T_2, then the experiment continues only until the m^{th} failure is observed, and all the remaining items are removed at the m^{th} failure time.

(iii) If the m^{th} failure time exceeds time T_2, then the experiment is terminated at time T_2 with removal of the remaining surviving items. Gòrny and Cramer (2016) assume that the total number of removed items including failure is represented by $\gamma_i = \begin{cases} \sum_{j=i}^{m}(R_j + 1), & i = 1, \dots, m \\ m + R_m - i + 1, & i = m+1, \dots, m + R_m \end{cases}$.

The count of observed failure till the time T_k is denoted by $D_k = \sum_{i=1}^{m+R_m} I_{(-\infty, T_k]}(X_{i:m+R_m:n}), k = 1,2$. The observed sample under above censoring design GTPH is represented as under:

Case (i) Occurrence of m^{th} failure before time T_1 is represented as $X_{m:m+R_m:n} \le X_{D_1:m+R_m:n} < T_1$;

then the sample is $X_{1:m+R_m:n}, \dots, X_{D_1:m+R_m:n}$, $D_1 \in (m, \dots, m + R_m)$

Case (ii) Occurrence of m^{th} failure time located between times T_1 and T_2 is represented as $T_1 \le X_{m:m+R_m:n} < T_2$;

then the sample is $X_{1:m+R_m:n}, \dots, X_{m:m+R_m:n}$, $D_1 \in (0, \dots, m-1), D_2 = m$

Case (iii) Occurrence of m^{th} failure time exceeding time T_2 is represented as $X_{m:m+R_m:n} > T_2$;

Then the sample is $X_{1:m+R_m:n}, \dots, X_{D_2:m+R_m:n}$, $D_1 \in (0, \dots, m-1)$

Likelihood functions L(θ) corresponding to the cases (i), (ii) and (iii) discussed above are given as (1), (2) and (3). L(θ) for the unknown parameter θ of a probability distribution with cumulative distribution function (*cdf*) denoted by $F(x)$ and probability density function (*pdf*) denoted by $f(x)$ is given as under,

$$L(\theta) = \begin{cases} \left[\prod_{i=1}^{d_1} \gamma_i\right] [1 - F(T_1;\theta)]^{\gamma_{d_1}+1} \prod_{i=1}^{d_1} f(x_{i:m+R_m:n};\theta) \left[1 - F(x_{i:m+R_m:n};\theta)\right]^{R_i} & (1) \\ \left[\prod_{i=1}^{m} \gamma_i\right] \prod_{i=1}^{m} f(x_{i:m+R_m:n};\theta) \left[1 - F(x_{i:m+R_m:n};\theta)\right]^{R_i} & (2) \\ \left[\prod_{i=1}^{d_2} \gamma_i\right] [1 - F(T_1;\theta)]^{\gamma_{d_2}+1} \prod_{i=1}^{d_2} f(x_{i:m+R_m:n};\theta) \left[1 - F(x_{i:m+R_m:n};\theta)\right]^{R_i} & (3) \end{cases}$$

III. THE NAKAGAMI DISTRIBUTION

A random variable X follows ND(α, λ) with shape parameter $\alpha \geq 0.5$ and scale parameter $\lambda > 0$, if its *pdf* is given by

$$f(x) = \frac{2}{\Gamma\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{(2\alpha-1)} \exp\left(-\frac{\alpha}{\lambda}x^2\right); \quad x > 0, \qquad (4)$$

The corresponding *cdf* is given by

$$F(x) = \frac{\Gamma(\frac{\alpha}{\lambda}x^2, \alpha)}{\Gamma\alpha}; \quad x > 0, \alpha \geq 0.5, \lambda > 0 \qquad (5)$$

such that $\Gamma(z, a) = \int_0^z t^{a-1} e^{-t} dt$ is the lower incomplete gamma function.

ND$(1, \lambda)$ is Rayleigh distribution, ND$(0.5, \lambda)$ is half-Normal distribution and if $y\sim$Gamma(α, θ) then $\sqrt{y}\sim$NGD$(\alpha, \alpha\theta)$

Reliability Characteristics

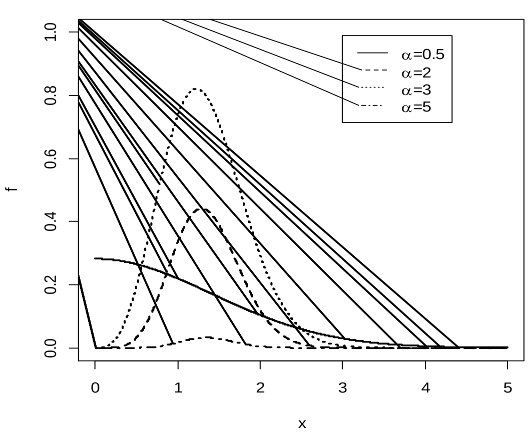
Mean time to system failure (MTSF): $\frac{\Gamma\alpha+0.5}{\Gamma\alpha}\sqrt{\frac{\lambda}{\alpha}}$; $\alpha \geq 0.5, \lambda > 0$

Reliability function: $1 - \frac{1}{\Gamma\alpha}\Gamma\left(\frac{\alpha}{\lambda}t^2, \alpha\right)$; $t > 0, \alpha \geq 0.5, \lambda > 0$

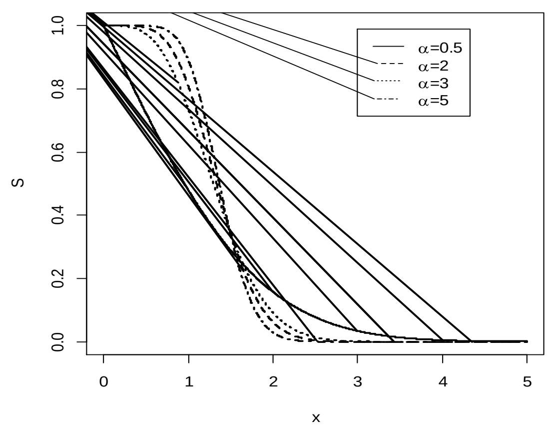
Mean residual life: $\frac{\int_u^\infty \left[1-\frac{1}{\Gamma\alpha}\Gamma\left(\frac{\alpha}{\lambda}t^2,\alpha\right)\right] dt}{\left[1-\frac{1}{\Gamma\alpha}\Gamma\left(\frac{\alpha}{\lambda}t^2,\alpha\right)\right]}$; $t > 0, \alpha \geq 0.5, \lambda > 0$

Hazard rate function: $\frac{\frac{2}{\Gamma\alpha}(\frac{\alpha}{\lambda})^{\alpha} t^{(2\alpha-1)} \exp\left(-\frac{\alpha}{\lambda}t^2\right)}{\left[1-\frac{1}{\Gamma\alpha}\Gamma\left(\frac{\alpha}{\lambda}t^2,\alpha\right)\right]}$; $t > 0, \alpha \geq 0.5, \lambda > 0$

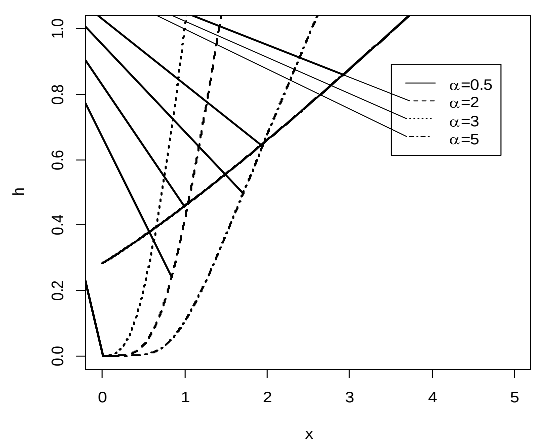
Fig. 1 illustrates flexibility of the ND model for adapting to the data variations in the shapes of *pdf*, *cdf*, reliability and hazard rate function of ND for different values of the parameter α while keeping value of the scale parameter λ as fixed.



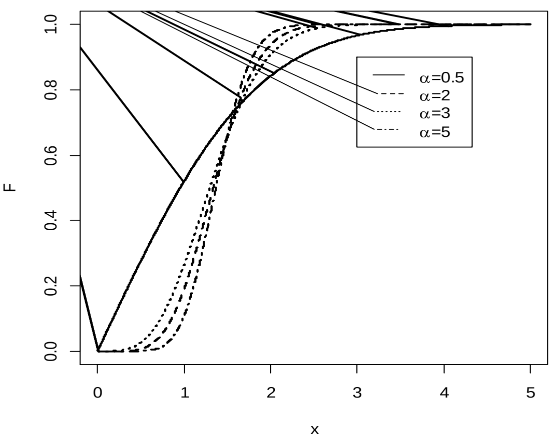
(a)



(b)



(c)



(d)

Fig.1: (a) Density f(x, α, λ) **(b)** Reliability function S(x, α, λ) **(c)** hazard rate function h(x, α, λ) **(d)** cumulative distribution function F(x, α, λ) of ND

Entropy

Entropy measures variation in the uncertainty thereby summarizing diversity in the data .

Renyi entropy is defined as $I_R(\delta) = \frac{1}{1-\delta} log\{\int_{-\infty}^{\infty} f(x)^{\delta} dx\}$, $\delta > 0$ and $\delta \neq 1$.

Hence, for ND : $I_R(\delta) = \log(2) + \frac{1}{2}\log(\alpha) + \frac{1}{1-\delta} log\left(\Gamma\alpha\delta - \frac{\delta-1}{2}\right) - \frac{\delta}{1-\delta} log(\Gamma\alpha) - \frac{\left(2\alpha\delta - \frac{\delta+1}{2}\right)}{1-\delta} log(\lambda) - \frac{\left(\alpha\delta - \frac{\delta-1}{2}\right)}{1-\delta} log(\delta)$ (6)

Shannon entropy is defined as $= E\left[-log\left(f(x)\right)\right]$. Hence, for ND :

$S = \frac{\alpha}{\lambda}E_1 - (2\alpha - 1)E_2 - log2 - log(\Gamma\alpha) - \alpha(\log(\alpha) + log\,(\lambda))$ (7)

such that $E_1 = E(x^2)$ and $E_2 = E\left(\log\,(x)\right)$

IV. Maximum Likelihood Estimation

For a GPTH sample $\underline{x} = (x_1, x_2, \ldots, x_k)$ from the ND, likelihood functions corresponding to (1)-(3) are specified as,

(i) $L(\alpha,\lambda|x) = c_1\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_1+1}} \left(\frac{2}{\Gamma\alpha}\right)^{d_1} \left(\frac{\alpha}{\lambda}\right)^{\alpha d_1} \prod_{i=1}^{d_1} x_i{}^{2\alpha-1} \left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i} \exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1} x_i{}^2\right)$

(ii) $L(\alpha,\lambda|x) = c_2\left(\frac{2}{\Gamma\alpha}\right)^m \left(\frac{\alpha}{\lambda}\right)^{\alpha m} \prod_{i=1}^m x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i} \exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^m x_i{}^2\right)$

(iii) $L(\alpha,\lambda|x) = c_3\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_2+1}} \left(\frac{2}{\Gamma\alpha}\right)^{d_1} \left(\frac{\alpha}{\lambda}\right)^{\alpha d_2} \prod_{i=1}^{d_2} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i} \exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2} x_i{}^2\right)$

where $c_1 = \prod_{i=1}^k \gamma_i$, $c_2 = \prod_{i=1}^k \gamma_i$ and $c_3 = \prod_{i=1}^k \gamma_i$

The corresponding log-likelihood functions are obtained as,

(i) $l(\alpha,\lambda) = ln(c_1) + \gamma_{d_1+1} ln\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha}{\Gamma\alpha}\right) + d_1(\ln(2) - \ln\,(\Gamma\alpha)) + d_1\alpha(\ln(\alpha) - \ln\,(\lambda)) + (2\alpha - 1)T_{11} - \frac{\alpha T_{12}}{\lambda} + \sum_{i=1}^{d_1} R_i \ln\,(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha})$,

(ii) $l(\alpha,\lambda) = ln(c_2) + m(\ln(2) - \ln\,(\Gamma\alpha)) + m\alpha(\ln(\alpha) - \ln\,(\lambda)) + (2\alpha - 1)T_{21} - \frac{\alpha T_{22}}{\lambda} \sum_{i=1}^m R_i \ln\,(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha})$,

(iii) $l(\alpha,\lambda) = ln(c_3) + \gamma_{d_2+1} ln\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha}{\Gamma\alpha}\right) + d_2(\ln(2) - \ln\,(\Gamma\alpha)) + d_2\alpha(\ln(\alpha) - \ln\,(\lambda)) + (2\alpha - 1)T_{31} - \frac{\alpha T_{32}}{\lambda} + \sum_{i=1}^{d_2} R_i \ln\,(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha})$,

where, $T_{i1} = \sum_{i=1}^k ln(x_i)$ and $T_{i2} = \sum_{i=1}^k x_i^2$, cases $i = 1,2,3$

Solutions of the following pairs of equations provide MLE$(\hat{\alpha}, \hat{\lambda})$ of the ND parameters(α, λ).

(i) $\frac{\partial l(\alpha,\lambda)}{\partial\alpha} = \gamma_{d_1+1} \frac{\left\{\Gamma\alpha' - \psi_{11}\left(\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha\right\}} - n\psi_1(\alpha) + d_1(\ln(\alpha) - \ln(\lambda) + 1) + 2T_{11} - \frac{T_{12}}{\lambda} + \sum_{i=1}^{d_1} R_i \frac{\left\{\Gamma\alpha' - \psi_{12}\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right\}}$

$\frac{\partial l(\alpha,\lambda)}{\partial\lambda} = -\gamma_{d_1+1} \frac{\left\{\phi_{11}\left(\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha\right\}} - \frac{d_1\alpha}{\lambda} + \frac{\alpha T_{12}}{\lambda^2} - \sum_{i=1}^{d_1} R_i \frac{\left\{\phi_{12}\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right\}}$ (8)

(ii) $\frac{\partial l(\alpha,\lambda)}{\partial\alpha} = -n\psi_1(\alpha) + m(\ln(\alpha) - \ln(\lambda) + 1) + 2T_{21} - \frac{T_{22}}{\lambda} + \sum_{i=1}^m R_i \frac{\left\{\Gamma\alpha' - \psi_{22}\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right\}}$

$\frac{\partial l(\alpha,\lambda)}{\partial\lambda} = -\frac{m\alpha}{\lambda} + \frac{\alpha T_{22}}{\lambda^2} - \sum_{i=1}^m R_i \frac{\left\{\phi_{22}\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right\}}$ (9)

(iii) $\frac{\partial l(\alpha,\lambda)}{\partial\alpha} = \gamma_{d_2+1} \frac{\left\{\Gamma\alpha' - \psi_{31}\left(\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha\right\}} - n\psi_1(\alpha) + d_2(\ln(\alpha) - \ln(\lambda) + 1) + 2T_{31} - \frac{T_{32}}{\lambda} + \sum_{i=1}^{d_2} R_i \frac{\left\{\Gamma\alpha' - \psi_{32}\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right\}}$

$\frac{\partial l(\alpha,\lambda)}{\partial\lambda} = -\gamma_{d_2+1} \frac{\left\{\phi_{31}\left(\frac{\alpha}{\lambda}T_2{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha\right\}} - \frac{d_2\alpha}{\lambda} + \frac{\alpha T_{32}}{\lambda^2} - \sum_{i=1}^{d_2} R_i \frac{\left\{\phi_{32}\left(\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)\right\}}{\left\{\Gamma\alpha - \Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right\}}$

(10)

such that $\psi(\alpha)=\frac{\partial \ln(\Gamma\alpha)}{\partial\lambda}$ $\quad \psi_{11}\left(\frac{\alpha}{\lambda}T_1{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha\right)}{\partial\alpha}$,

$\psi_{12}\left(\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)}{\partial\alpha}$

$\psi_{22}\left(\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)}{\partial\alpha}$, $\quad \psi_{31}\left(\frac{\alpha}{\lambda}T_2{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha\right)}{\partial\alpha}$,

$\psi_{32}\left(\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)}{\partial\alpha}$

$\emptyset_{11}\left(\frac{\alpha}{\lambda}T_1{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha\right)}{\partial\lambda}$, $\quad \emptyset_{12}\left(\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)}{\partial\lambda}$

$\emptyset_{22}\left(\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)}{\partial\lambda}$

$\emptyset_{31}\left(\frac{\alpha}{\lambda}T_2{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha\right)}{\partial\lambda}$, $\quad \emptyset_{32}\left(\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)=\frac{\partial\left(\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha\right)}{\partial\lambda}$,

$\Gamma\alpha' =\frac{\partial\Gamma\alpha}{\partial\alpha}$

MLEs are obtained in mathematically intractable form, hence numerical solution for (ii) are obtained using Newton-Raphson (NR) iterative method for given values of $(n,m,\underline{R},\underline{x})$. Similarly MLE*s* under (i) and (iii) are evaluated given $(T_1,d_1,n,\underline{R},\underline{x})$ and $(T_2,d_2,n,\underline{R},\underline{x})$ respectively.

Using invariance property of MLE*s* we get,

$M\hat{T}SF=\frac{\Gamma\hat{\alpha}+0.5}{\Gamma\hat{\alpha}}\sqrt{\left(\frac{\hat{\alpha}}{\hat{\lambda}}\right)}$

$\widehat{R}(t)=1-\frac{1}{\Gamma\hat{\alpha}}\Gamma\left(\frac{\hat{\alpha}}{\hat{\lambda}}t^2,\hat{\alpha}\right)\;;\;\; t>0$

$$\widehat{h}(t)=\frac{\frac{2}{\Gamma\hat{\alpha}}\left(\frac{\hat{\alpha}}{\hat{\lambda}}\right)^{\hat{\alpha}}t^{2\hat{\alpha}-1}\exp\left(-\frac{\hat{\alpha}}{\hat{\lambda}}t^2\right)}{1-\frac{1}{\Gamma\hat{\alpha}}\Gamma\left(\frac{\hat{\alpha}}{\hat{\lambda}}t^2,\hat{\alpha}\right)}\;;\quad t>0$$

V. Bayesian Estimation

Bayesian parameter estimation under symmetric SELF, weighs losses for over estimation and under estimation equally and is equal to the posterior mean. Asymmetric linear exponential loss function (LINEX) introduced by Varian (1975) is defined by $L(\delta)\propto e^{c\delta}-c\delta-1, c\neq 0$, where $\delta=\hat{\theta}-\theta$. The sign and magnitude of *c* reflects the direction and degree of asymmetry, respectively.

Bayes estimate under SELF and under LINEX is given by $\hat{\theta}_L=E_\theta(\theta)$ and $\hat{\theta}_L=-\frac{1}{c}lnE_\theta\left(e^{-c\theta}\right)$ respectively, provided that $E_\theta(\theta)$ and $E_\theta\left(e^{-c\theta}\right)$ exists and is finite, where E_θ denotes the expectation relative to the posterior distribution.

Estimation under known shape parameter

Under the assumption that the parameter α *is known*, the natural family of conjugate prior for the scale parameter λ is given by $g(\lambda)=\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}\;;\;\lambda>0,d>0,\vartheta>0$. Prior *mean* and *variance* of the unknown parameter λ are summarized as $\frac{\vartheta}{\delta}$ and $\frac{\vartheta}{\delta^2}$ respectively. Posterior *mean* for the three cases (i), (ii) and (iii) :

Under SELF

(i) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{s}}=\frac{1}{k_1}\int_0^\infty\lambda\left(1-\frac{\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_1+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_1}\prod_{i=1}^{d_1}x_i{}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1}x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda$$

where $$k_1=\int_0^\infty\left(1-\frac{\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_1+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_1}\prod_{i=1}^{d_1}x_i{}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1}x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda$$

(ii) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{s}}=\frac{1}{k_2}\int_0^\infty\lambda\left(\frac{2}{\Gamma\alpha}\right)^m\left(\frac{\alpha}{\lambda}\right)^{\alpha m}\prod_{i=1}^{m}x_i{}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{m}x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda$$ where

$$k_2=\int_0^\infty\left(\frac{2}{\Gamma\alpha}\right)^m\left(\frac{\alpha}{\lambda}\right)^{\alpha m}\prod_{i=1}^{m}x_i{}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{m}x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda$$

(iii) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{s}}=\frac{1}{k_3}\int_0^\infty\lambda\left(1-\frac{\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_2+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_2}\prod_{i=1}^{d_2}x_i{}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2}x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda$$

where

$$k_3=\int_0^\infty\left(1-\frac{\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_2+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_2}\prod_{i=1}^{d_2}x_i{}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2}x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda$$

(11) - (13)

Under LINEX

(i) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{L}}=-\frac{1}{c}ln\left(\frac{1}{k_1}\int_0^\infty e^{-c\lambda}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_1+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_1}\prod_{i=1}^{d_1}x_i{}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1}x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda\right)$$

(ii) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{L}}=-\frac{1}{c}ln\left(\frac{1}{k_2}\int_0^\infty e^{-c\lambda}\left(\frac{2}{\Gamma\alpha}\right)^m\left(\frac{\alpha}{\lambda}\right)^{\alpha m}\prod_{i=1}^{m}x_i{}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{m}x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda\right)$$

(iii) $\hat{\boldsymbol{\lambda}}_{\boldsymbol{L}} = -\frac{1}{c} ln\left(\frac{1}{k_3}\int_0^\infty e^{-c\lambda}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma d_2+1}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_2}\prod_{i=1}^{d_2} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2} x_i{}^2\right)\frac{d^\vartheta}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-d\lambda}d\lambda\right)$

(14) - (16)

For given $(T_1, d_1, n, \underline{R}, \underline{x})$ for case (i), given $(n, m, \underline{R}, \underline{x})$ for case (ii) and given $(T_2, d_2, n, \underline{R}, \underline{x})$ for case (iii), Bayes estimators of λ *given* α under SELF are obtained as (11) - (13) and under LINEX loss function are obtained as (14) - (16).

Estimation under both parameters unknown

Conditional prior distribution of λ *given* α, appropriately represented by the conjugate gamma density function with hyper-shape parameter $\vartheta > 0$ is assumed as $g_1(\lambda|\alpha) = \frac{\alpha^{-\vartheta}}{\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{\lambda}{\alpha}}$; $\lambda > 0$. Decaying exponential prior distribution indicating fatigue is assumed for the shape parameter α represented as $g_2(\alpha) = \frac{1}{d}e^{-\frac{1}{d}\left(\alpha-\frac{1}{2}\right)}$; $\alpha > 0.5$, where d is the positive hyper-scale parameter.

Joint bivariate prior density of (α, λ) is given as, $(\alpha, \lambda) = \frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}$; $\alpha > 0.5, \lambda > 0$. Posterior *mean* for the three cases (i), (ii) and (iii) :

Bayes estimator of α under SELF

(i) $\hat{\boldsymbol{\alpha}}_{\boldsymbol{s}} = \int_{0.5}^\infty \alpha\int_0^\infty \frac{1}{h_1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma d_1+1}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_1}\prod_{i=1}^{d_1} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha$

where $h_1 = \iint\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma d_1+1}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_1}\prod_{i=1}^{d_1} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha$

(ii) $\hat{\boldsymbol{\alpha}}_{\boldsymbol{s}} = \int_{0.5}^\infty \alpha\int_0^\infty \frac{1}{h_2}\left(\frac{2}{\Gamma\alpha}\right)^{m}\left(\frac{\alpha}{\lambda}\right)^{\alpha m}\prod_{i=1}^{m} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{m} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha$

where $h_2 = \iint\left(\frac{2}{\Gamma\alpha}\right)^{m}\left(\frac{\alpha}{\lambda}\right)^{\alpha m}\prod_{i=1}^{m} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{m} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha$

(iii) $\hat{\boldsymbol{\alpha}}_{\boldsymbol{s}} = \int_{0.5}^\infty \alpha\int_0^\infty \frac{1}{h_3}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma d_2+1}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_2}\prod_{i=1}^{d_2} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha$

where $h_3 = \iint\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma d_2+1}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_2}\prod_{i=1}^{d_2} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha$

(17) - (19)

Bayes estimator of α under LINEX loss function

(i) $\hat{\boldsymbol{\alpha}}_{\boldsymbol{L}} = -\frac{1}{c} ln\left(\int_{0.5}^\infty e^{-c\alpha}\int_0^\infty \frac{1}{h_1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_1{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma d_1+1}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_1}\prod_{i=1}^{d_1} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha\right)$

(ii) $\hat{\boldsymbol{\alpha}}_{\boldsymbol{L}} = -\frac{1}{c} ln\left(\int_{0.5}^\infty e^{-c\alpha}\int_0^\infty \frac{1}{h_2}\left(\frac{2}{\Gamma\alpha}\right)^{m}\left(\frac{\alpha}{\lambda}\right)^{\alpha m}\prod_{i=1}^{m} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{m} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha\right)$

(iii) $\hat{\boldsymbol{\alpha}}_{\boldsymbol{L}} = -\frac{1}{c} ln\left(\int_{0.5}^\infty e^{-c\alpha}\int_0^\infty \frac{1}{h_3}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}T_2{}^2,\alpha}{\Gamma\alpha}\right)^{\gamma d_2+1}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_2}\prod_{i=1}^{d_2} x_i{}^{2\alpha-1}\left(1 - \frac{\Gamma\frac{\alpha}{\lambda}x_i{}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2} x_i{}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\lambda d\alpha\right)$

(20-22)

Bayes estimator of λ under SELF

(i) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{s}} = \int_0^\infty \lambda \int_{0.5}^\infty \frac{1}{h_1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{T_1}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_1+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_1}\prod_{i=1}^{d_1}{x_i}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{x_i}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1}{x_i}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\alpha d\lambda$$

(ii) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{s}} = \int_0^\infty \lambda \int_{0.5}^\infty \frac{1}{h_2}\left(\frac{2}{\Gamma\alpha}\right)^{m}\left(\frac{\alpha}{\lambda}\right)^{\alpha m}\prod_{i=1}^{m}{x_i}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{x_i}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{m}{x_i}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\alpha d\lambda$$

(iii) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{s}} = \int_0^\infty \lambda \int_{0.5}^\infty \frac{1}{h_3}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{T_2}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_2+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_2}\prod_{i=1}^{d_2}{x_i}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{x_i}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2}{x_i}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\alpha d\lambda$$

(23-25**)**

Bayes estimator of λ under LINEX loss function

(i) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{L}} = -\frac{1}{c}ln\left(\int_0^\infty e^{-c\lambda}\int_{0.5}^\infty \frac{1}{h_1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{T_1}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_1+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_1}\prod_{i=1}^{d_1}{x_i}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{x_i}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_1}{x_i}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\alpha d\lambda\right)$$

(ii) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{L}} = -\frac{1}{c}ln\left(\int_0^\infty e^{-c\lambda}\int_{0.5}^\infty \frac{1}{h_2}\left(\frac{2}{\Gamma\alpha}\right)^{m}\left(\frac{\alpha}{\lambda}\right)^{\alpha m}\prod_{i=1}^{m}{x_i}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{x_i}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{m}{x_i}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\alpha d\lambda\right)$$

(iii) $$\hat{\boldsymbol{\lambda}}_{\boldsymbol{L}} = -\frac{1}{c}ln\left(\int_0^\infty e^{-c\lambda}\int_{0.5}^\infty \frac{1}{h_3}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{T_2}^2,\alpha}{\Gamma\alpha}\right)^{\gamma_{d_2+1}}\left(\frac{2}{\Gamma\alpha}\right)^{d_1}\left(\frac{\alpha}{\lambda}\right)^{\alpha d_2}\prod_{i=1}^{d_2}{x_i}^{2\alpha-1}\left(1-\frac{\Gamma\frac{\alpha}{\lambda}{x_i}^2,\alpha}{\Gamma\alpha}\right)^{R_i}\exp\left(-\frac{\alpha}{\lambda}\sum_{i=1}^{d_2}{x_i}^2\right)\frac{\alpha^{-\vartheta}}{d\Gamma\vartheta}\lambda^{\vartheta-1}e^{-\frac{1}{d}\left(\frac{\alpha^2+d\lambda}{\alpha}-\frac{1}{2}\right)}d\alpha d\lambda\right)$$

(26-28)

For given $(T_1, d_1, n, \underline{R}, \underline{x})$ for (i), given $(n, m, \underline{R}, \underline{x})$ for (ii) and given $(T_2, d_2, n, \underline{R}, \underline{x})$ for (iii), we obtain Bayes estimators of the unknown parameters α and λ under SELF and LINEX loss function.

VI. Simulation Study

In the previous sections, we have obtained expressions for MLE*s* and for Bayes estimators under SELF and LINEX loss function of shape (α) and scale (λ) parameters for ND under GTPH. Objective of the simulation study is to assert that Bayes estimates are always closer to the true values vis-a vis MLE*s*.

In this section, we present Markov Chain Monte Carlo (MCMC) study to observe the performance of inferential procedures developed in the previous sections for different sample sizes and under different choices of censoring ratios. All the simulation work has been undertaken using statistical software R.

MCMC algorithm is stated as under:

(i) We generate random samples of size *n=20*, *30* and *50* for GTPH schemes for case (ii) using methodology proposed by Balakrishnan and Sandhu (1995).

(ii) We compute MLE*s* $\hat{\alpha}_M$ and $\hat{\lambda}_M$ of parameter λ and α under different sample sizes and for different proposed censoring scheme with true values of $(\alpha, \lambda) = (2,1)$ and $(\alpha, \lambda) = (3,2)$ by NR method.

(iii) We compute Bayes estimators $\hat{\alpha}_S$, $\hat{\lambda}_S$, $\hat{\alpha}_L$ and $\hat{\lambda}_L$ for the parameters α and λ under SELF and LINEX for different sample sizes *n* and under different three specifications of random removals for each combination of (*n* , *m*) censoring scheme, assuming the initial values of $(\alpha, \lambda) = (2,1)$ and $(\alpha, \lambda) = (3,2)$ with hyper parameter values fixed at *d=3*, *ϑ=4* and *δ=4* .

(iv) Steps 1-3 are repeated *N=10000* times. Means and MSEs are computed for the generated Monte Carlo samples of different sample sizes *n* and also for the effective sample sizes *m*.

(
2

Table 1: Estimated means and MSEs of parameter α for different censoring ratios under GPH for case (ii) when true value of $\alpha=3$

n	*m*	***Censoring scheme***	$\hat{\alpha}_M$	$MSE(\alpha_M)$	$\hat{\alpha}_S$	$MSE(\alpha_S)$	$\hat{\alpha}_L$	$MSE(\alpha_L)$
20	8	(2*3, 0*2, 2*3)	2.6744	0.8774	2.4656	0.5221	2.2736	0.3933
	12	(4*1, 0*10, 4*1)	3.7523	0.9089	2.1582	0.3617	2.4721	0.3043
	16	(2*1, 0*14, 2*1)	2.8952	0.8053	2.8849	0.4474	3.1428	0.2747
30	12	(3*3, 0*6, 3*3)	3.4596	0.7492	2.4116	0.3617	3.5362	0.1944
	18	(2*3, 0*12, 2*3)	3.5496	0.6197	2.5421	0.2616	3.2571	0.2256
	24	(1*3, 0*18, 1*3)	3.2325	0.5759	2.7071	0.3326	2.7072	0.0874
50	20	(5*3, 0*14, 5*3)	3.3617	0.3562	2.7896	0.0873	2.7078	0.1808
	30	(2*5, 0*20, 2*5)	3.1213	0.4372	2.8045	0.1672	3.0963	0.0873
	40	(5*1, 0*38, 5*1)	3.0321	0.1935	2.9651	0.0583	3.0312	0.0776

Table 2: Estimated means and MSEs of parameter λ for different censoring ratios under GPH for case (ii) if true value of $\lambda=1$

n	*m*	***Censoring scheme***	$\hat{\lambda}_M$	MSE(λ_M)	$\hat{\lambda}_S$	MSE(λ_S)	$\hat{\lambda}_L$	MSE(λ_L)
20	8	(2*3, 0*2, 2*3)	0.8638	0.0583	1.4387	0.1846	1.5045	0.1846
	12	(4*1, 0*10, 4*1)	0.8714	0.0369	0.7206	0.1106	1.4823	0.1523
	16	(2*1, 0*14, 2*1)	0.9452	0.0339	0.8257	0.0897	0.8978	0.1106
30	12	(3*3, 0*6, 3*3)	1.3417	0.0342	1.2915	0.0459	1.2915	0.1178
	18	(2*3, 0*12, 2*3)	1.2697	0.0419	1.3118	0.0507	1.2532	0.0897
	24	(1*3, 0*18, 1*3)	0.8956	0.0291	0.8798	0.0293	0.8535	0.0221
50	20	(5*3, 0*14, 5*3)	0.9376	0.0175	1.2374	0.0243	1.2394	0.0548
	30	(2*5, 0*20, 2*5)	0.9858	0.0135	1.0472	0.0321	1.2111	0.0375
	40	(5*1, 0*38, 5*1)	0.9921	0.0211	1.0233	0.0128	0.9921	0.0129

Table 3: Estimated means and MSE*s* of parameter *α* for different censoring ratios under GPH for case (ii) when true value of *α=2*

n	*m*	***Censoring scheme***	$\hat{\lambda}_M$	MSE(λ_M)	$\hat{\lambda}_S$	MSE(λ_S)	$\hat{\lambda}_L$	MSE(λ_L)
20	8	(2*3, 0*2, 2*3)	1.5632	0.1935	1.4113	0.1949	1.5923	0.1893
	12	(4*1, 0*10, 4*1)	1.6532	0.1132	1.6137	0.1281	2.1567	0.1195
	16	(2*1, 0*14, 2*1)	1.7806	0.0748	2.3379	0.1032	1.7364	0.1089
30	12	(3*3, 0*6, 3*3)	2.3475	0.0574	1.7635	0.1892	2.2672	0.0879
	18	(2*3, 0*12, 2*3)	1.7581	0.0676	1.7945	0.0543	1.7673	0.0675
	24	(1*3, 0*18, 1*3)	2.2197	0.0484	1.9754	0.0635	2.1567	0.0524
50	20	(5*3, 0*14, 5*3)	1.8784	0.0518	2.2378	0.0512	2.1134	0.0423
	30	(2*5, 0*20, 2*5)	1.9725	0.0336	1.9831	0.0448	2.2168	0.0394
	40	(5*1, 0*38, 5*1)	1.9830	0.0145	2.1193	0.0112	1.9832	0.0129

Table 4: Estimated means and MSE*s* of parameter *λ* for different censoring ratios under GPH for case (ii) when true value of *λ=2*

n	*m*	***Censoring scheme***	$\hat{\alpha}_M$	MSE(α_M)	$\hat{\alpha}_S$	MSE(α_S)	$\hat{\alpha}_L$	MSE(α_L)
20	8	(2*3, 0*2, 2*3)	2.2527	0.4053	1.4722	0.3697	1.7652	0.3697
	12	(4*1, 0*10, 4*1)	1.7765	0.3664	1.6691	0.1767	1.8223	0.3189
	16	(2*1, 0*14, 2*1)	1.7953	0.3422	1.8221	0.2043	2.4527	0.2043
30	12	(3*3, 0*6, 3*3)	2.2161	0.1859	2.5123	0.1351	1.8228	0.2359
	18	(2*3, 0*12, 2*3)	2.1976	0.1971	2.6043	0.1016	2.6049	0.1832
	24	(1*3, 0*18, 1*3)	1.9821	0.1725	1.8750	0.0868	2.5367	0.0663
50	20	(5*3, 0*14, 5*3)	1.9091	0.1253	2.2516	0.0663	2.4523	0.1767
	30	(2*5, 0*20, 2*5)	1.9623	0.0931	2.1431	0.0576	2.2871	0.0676
	40	(5*1, 0*38, 5*1)	2.1168	0.0526	2.0734	0.0433	2.1255	0.0342

Table 5: LL*s*, -2lnL, AIC, BIC, K-S and AD of the fitted distribution for $\mathbf{D_1}$.

Distribution	***Parameters Estimates***		***-2lnL***	***AIC***	***BIC***	***K-S***	***AD***
	A	***λ***					
Nakagami	4.8336	5.2823	623.8586	627.8586	632.0137	0.0514	0.1596
Gamma	18.0975	2.5927	1985.681	1989.681	1993.836	0.0664	0.2335
Weibull	4.6488	7.6130	726.08	730.08	734.235	0.0662	0.2843

Table 6: LL*s*, -2lnL, AIC, BIC, K-S and AD of the fitted distribution for **D_2**.

Distribution	***Parameters Estimates***		***-2lnL***	***AIC***	***BIC***	***K-S***	***AD***
	α	λ					
Nakagami	6.2229	6.2506	182.6428	186.6428	191.111	0.0542	0.2446
Gamma	23.3808	9.5380	590.8718	594.8718	599.34	0.0696	0.3792
Weibull	5.5048	2.6508	253.6034	257.6034	262.0717	0.0567	0.2568

Table 7: Estimated means and MSE*s* of parameter α for different censoring ratios under GPH of case (ii) for **D_1**

n	*M*	***Censoring scheme***	$\hat{\alpha}_M$	MSE($\hat{\alpha}_M$)	$\hat{\alpha}_S$	MSE($\hat{\alpha}_S$)	$\hat{\alpha}_L$	MSE($\hat{\alpha}_L$)
20	8	(2*3, 0*2, 2*3)	5.9118	0.0994	5.9001	0.0847	5.9138	0.0797
	12	(4*1, 0*10, 4*1)	5.9162	0.0926	5.9239	0.0845	5.9155	0.0659
	16	(2*1, 0*14, 2*1)	5.9399	0.0749	5.9297	0.0696	5.9399	0.0565
30	12	(3*3, 0*6, 3*3)	5.9166	0.0890	5.9278	0.0803	5.9166	0.0646
	18	(2*3, 0*12, 2*3)	5.9360	0.0686	5.9570	0.0639	5.9381	0.0510
	24	(1*3, 0*18, 1*3)	5.9511	0.0502	5.9707	0.0439	5.9552	0.0353
50	20	(5*3, 0*14, 5*3)	5.9438	0.0496	5.9631	0.0520	5.9683	0.0490
	30	(2*5, 0*20, 2*5)	5.9515	0.0313	5.9892	0.0387	5.9853	0.0302
	40	(5*1, 0*38, 5*1)	5.9718	0.0216	5.9895	0.0153	5.9972	0.0162

Table 8: Estimated means and MSE*s* of parameter α for different censoring ratios under GPH of case (ii) for **D_2**.

n	*M*	*Censoring Scheme*	$\hat{\alpha}_M$	MSE($\hat{\alpha}_M$)	$\hat{\alpha}_S$	MSE(α_S)	α_L	MSE(α_L)
20	8	(2*3, 0*2, 2*3)	4.9058	0.0958	4.9197	0.0984	4.9331	0.0723
	12	(4*1, 0*10, 4*1)	4.9356	0.0613	4.9265	0.0639	4.9339	0.0594
	16	(2*1, 0*14, 2*1)	4.9383	0.0478	4.9389	0.0537	4.9395	0.0300
30	12	(3*3, 0*6, 3*3)	4.9462	0.0651	4.9266	0.0797	4.9381	0.0682
	18	(2*3, 0*12, 2*3)	4.9581	0.0579	4.9400	0.0354	4.9564	0.0388
	24	(1*3, 0*18, 1*3)	4.9732	0.0463	4.9527	0.0286	4.9683	0.0127
50	20	(5*3, 0*14, 5*3)	4.9624	0.0538	4.9407	0.0379	4.9566	0.0213
	30	(2*5, 0*20, 2*5)	4.9760	0.0356	4.9703	0.0265	4.9742	0.0127
	40	(5*1, 0*38, 5*1)	4.9824	0.0216	4.9829	0.0120	4.9970	0.0113

Posterior distributions obtained in section 5 are not in tractable form. Hence, we use MCMC approximation technique to obtain the parameter estimates via simulation. Simulation study shows that there is a decrease in MSE*s* as we increase the sample size, hence increase in sample size increases accuracy of the estimates obtained (**Table 1-4**).

Also, Bayes estimates obtained under LINEX have smaller MSE*s* than those under SELF while Bayes estimates record smaller MSE*s* compared to MLE*s*. Bayes estimates under

LINEX loss are found to be more precise (**Table 1-4**) and thus superior to the conventional MLE*s* for situations that confirm to ND pattern of hazard curve.

VII. Real Data Analysis

This section demonstrates application of ND for lifetime modelling of electronic devices (**D_1**) and hardware such as fibres (**D_2**) using two real data sets. *Fig. 2* and *Fig. 3* display density, distribution and reliability characteristics graphically for the ND fitted data sets. For comparative assessment of better fit, popular lifetime distributions such as gamma and Weibull distributions are also fitted to the same data sets. Best model among the three is determined based on the log-likelihood (LL) measure, Kolmogorov-Smirnov (K-S) test, Akaike Information Criterion (AIC), Bayesian Information Criteria (BIC) and Anderson-Darling statistic (AD) which are given in **Table 5-6.** Computed values of MLE*s*($\hat{\alpha}_M, \hat{\lambda}_M$), and Bayes estimates under SELF($\hat{\alpha}_S, \hat{\lambda}_S$) and Linex loss($\hat{\alpha}_L, \hat{\lambda}_L$) are presented in **Table 7-8** for the following two real lifetime data sets **D_1** and **D_2** :

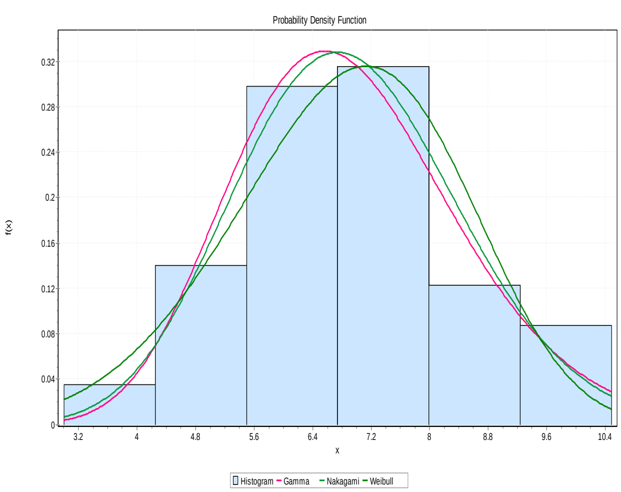
D_1 (Data Set 1): Data reported by Schafft *et al*. (1987) and taken from Lawless (2003) represents hours to failure of 59 conductors of 400-micrometer length. All reported test units failed at the same given high temperature and current density environment, thus providing homogenous conditions for concomitant variables.

D_2 (Data Set 2): Data representing tensile strength of 69 carbon fibres (measured in GPa) which were tested under tension at gauge lengths of 20mm were taken from Bader and Priest (1982).

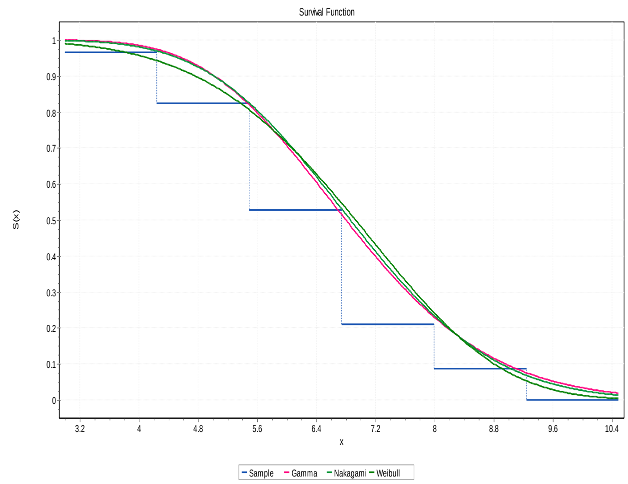
The formulae for computing AIC and BIC are as follows:

$\mathrm{AIC} = -2\ln\mathrm{L} + 2k$ and $\mathrm{BIC} = -2\ln\mathrm{L} + k\log(\mathrm{n})$, where $k =$ the number of parameters and n represents the sample size.

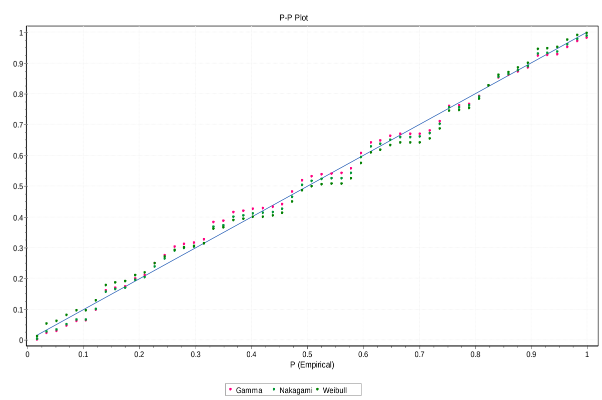
(a)



(b)



(c)



(d)

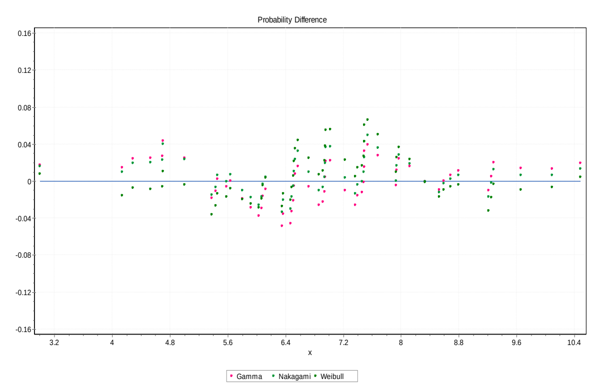


Fig. 2: (a)The fitted density functions of Gamma, Nakagami and Weibull superimposed with empirical histogram,
(b) respective reliability functions,
(c) respective probability difference plots and
(d) respective PP plots of sample data

VIII. CONCLUSION

This article presents a new and better fitted lifetime distribution for reliability analysis of machinery components and industrial wares. We present mathematical properties and develop expressions for the classical and the Bayes estimators for ND under GTPH. Hazard curve of ND shows that it can be used to model robust items with long lifetimes. GTPH ensures a certain minimum failure observations while ensuring that the experimental time is not overly prolonged. Intermittent removals of live test units under progressive mode of censoring provide a strategic means of limiting the experimental time for robust items under life-test. Parameter estimation case when one parameter is known has been presented, followed by developing theoretical framework for the case of both parameters unknown - under the classical as well as Bayes paradigm. Descriptive numerical assessment based on a simulation study and two empirical data based studies is carried out. MLE, and Bayes estimators under a symmetric and an asymmetric loss function individually are developed. For the chosen data sets from the classical life testing experimental trials, ND has been *adjudged best fit according to deviance summaries and other performance indicators of goodness of fit*, both for the complete data set for the classical MLE (Tables 5 and 6), as well as under various censoring scenarios considered under GTPH for the Bayes estimates derived from posterior distributions (Tables 7 and 8). The final objective is to discover a new model approach to analyse industrial output data and to advocate use of the more efficient Bayes estimation strategy for life-test experiments. Thus, our present study favours ND as a strong contender for lifetime modelling of machine components vis-a-vis the conventional gamma and Weibull models.

Declaration Authors declare that there is no conflict of interest.

FUNDING ACKNOWLEDGEMENT

The first author gratefully acknowledges FRP grant from IoE, University of Delhi.

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