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Cooperative Behaviour of Fitzhugh-Nagumo System Under Direct-Indirect Coupling

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Abstract: **Many significant phenomenon like Synchronisation, suppression of oscillation are observed in coupled nonlinear dynamical systems. Here I investigate on two diffusively coupled FitzHugh-Nagumo system (which are in self sustained periodic firing mode), in presence of environment coupling and notice they collectively transit to amplitude death states from oscillatory states. By controlling the coupling parameters the coupled system goes to amplitude death state through inverse Hopf bifurcation. Transition from oscillation to amplitude death state in different parametric zone(two-parameter space) is studied here. Though direct-indirect coupling is general methode of inducing amplitude death but this Study shows depending on parametric condition the system can revive its oscillatory states. Theoretical linear stability analysis and numerical results for different transition scenarios in different paranetric zone agree with each other. The route of transition between oscillation and amplitude death state is studied through bifurcation plot using XPPAUT AUTO package.**

Index Terms: **Amplitude death, synchronisation, hopf bifurcation, periodic firing, diffusive coupling, environment coupling, revival of oscillation.**

I. Introduction

In recent time the researchers in physical, chemical and biological science are deeply interested and engaged to study on collective dynamics of coupled nonlinear systems like synchronisation, amplitude death, oscillation death, chimera etc (Glass, 2001; Saxena, 2012; Banerjee, 2013, 2018). One of important manifestation of suppression of oscillation is amplitude death (AD), which occurs in coupled systems when they mutually interact and collectively go to a stable homogeneous steady state (HSS) by suppressing their oscillation under some parametric conditions (Saxena, 2012; Prasad, 2005; Reddy, 1998). Various coupling scheme like mean field coupling (Sharma, 2012), environmental coupling (Resmi, 2011, 2012; Biswas, 2017), conjugate coupling (Karnatak, 2007), nonlinear coupling (Prasad, 2010), with application of local filter (Banerjee, 2018) etc. has been proposed to suppress the oscillation in coupled sytem. Among them the direct-indirect coupling scheme (Resmi, 2011) is important to induce amplitude death. Though it was proposed to induce AD but later with this coupling scheme another important type of oscillation quenching, called oscillation death, is observed in reference (Ghosh, 2014). Environmental coupling is quite natural in many natural system (Goldbeter, 1996; Kruse, 2005; Arumugam, 2016). The applications of AD has many aspects like neuronal systems (Ermentrout, 1990), the suppression of unwanted fluctuations in lasers (Kumar, 2008), electronic circuits (Banerjee, 2013; Biswas, 2018) etc.

Neurons, a basic information processing unit, can generate electrical signals, called action potential, in response to chemical and other inputs, and capable to transmit them to other cells through synaptic connections (junction point of two neurons). So neurons interact directly through ion diffusion and indirectly through their common environment.

Neurons are not isolated entities. They form a complex networks spanning the entire body and the communication between neurons is done through travelling electrical signals known as action potential, which are governed by electrophysiological properties of the cell. Typical neurons may make up to 10,000 synaptic contacts with other neurons (Feng, 2004) or occasionally over 100,000 (Feng, 2004; Izikevich, 2007). So understanding of coupled behaviour of neuronal system is important for both academic and application point of view.

To study the collective dynamics of neuronal system under the simultaneous presence of both diffusive (direct) and environment(indirect) coupling I consider the well known and most widely used neuronal model FitzHugh-Nagumo model(FHN) (FitzHugh, 1961; Nagumo, 1962). In 1960 Richard FitzHugh and later J. Nagumo introduced a model based on Hodgkin-Huxley model (Hodgkin, 1952), known as FitzHugh-

Nagumo model which very successfully modelled the initiation and propagation of neural action potential which are also known as firing of a spike.

This paper represents my study on the collective dynamics of two FHN systems, in periodic firing mode, in presence of direct-indirect coupling. I observe that coupling coefficients control the dynamics and systems enter into AD State through inverse Hopf bifurcation. I also notice revival of oscillation may occur in some parametric condition. Use of XPPAUT 6.1. recognise the bifurcation points are Hopf points. I also have done stability analysis to get the necessary conditions at which transition is occurred. The numerical study agrees well with this theoretical analysis.

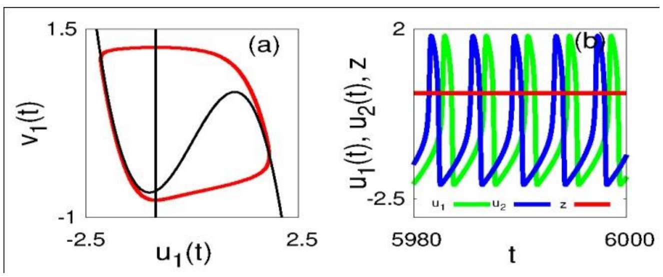

Fig. 1. (a) Phase diagram for individual FHN system (here system 1), Black lines are nullclines. (b) time series of u_1 and u_2 (periodic firing mode of activators) of uncoupled FHN system and z (environment).

The rest of the paper is organized in the following manner: The mathematical model of diffusively coupled two FHN systems. The systems are also coupled through a common environment. This is described in Section II. The necessary theoretical stability analysis is done in Section III. The numerical results are represented in Section IV. After summarization of main findings the conclusion of the whole study is described in Section V.

II. System description

Here I consider two identical FitzHugh-Nagumo(FHN) system (FitzHugh, 1961; Nagumo, 1962; Lindner, 2004) in self sustained periodic firing mode coupled diffusively as well as indirectly through a common environment z. To want to see the dynamics of this coupled system here I used the generalized model proposed in reference (Resmi, 2011) where two identical oscillators are coupled diffusively and at the same time they are also coupled through a common environment z. This environment is modeled using an over-damped oscillator with damping coefficient k (> 0). With theses FHN systems The mathematical model gets the shape as equations (1) given by

$$
\begin{aligned}
r\dot{u_1} &= u_1 - \frac{u_1^3}{3} - v_1 + d(u_2 - u_1) + \varepsilon z \\
\dot{v_1} &= u_1 + a \\
r\dot{u_2} &= u_2 - \frac{u_2^3}{3} - v_2 + d(u_1 - u_2) + \varepsilon z \\
v_2 &= u_2 + a \\
\dot{z} &= -kz - \varepsilon \frac{(u_1 + u_2)}{2}
\end{aligned}
\tag{1}
$$

Where $u_i$and v_i activator and inhibitor variable of the system $(i = 1{,}2)$. d and ε represents diffusive coupling constant and environmental coupling constant respectively. In the absence of both u_i-systems, the environment decays monotonically towards the zero steady state and remains in that dormant state. In the uncoupled FHN system i.e. when d = 0 and ε = 0, the parameter a and r control the system dynamics. Due to a supercritical Hopf bifercation at $|a| = 1$ the systen exhibits self sustained periodic firing mode for $|a| > 1$ while for $|a| < 1$ it is excitable. The parameter r controls the time scale in the system. In this paper I use the value of $a = 0.85, r = 0.1$which confirm that the each isolated FHN systems are in periodic firing mode.

In the right of the Fig. 1 shows the periodic firing mode(green line for u_1 , blue for u_2 and red for z) of uncoupled system, phase diagram of system 1(u_1 vs v_1 graph)(in left). During my study, interestingly, I notice that the systems go to Amplitude Death state for different parameter value d, ε and k.

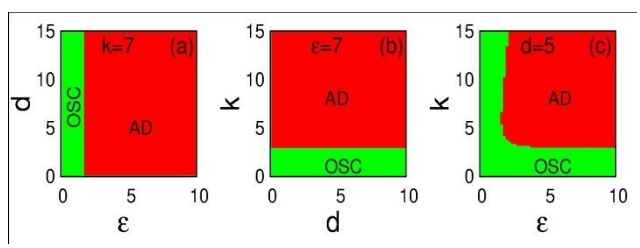

Fig. 2. From theoretical analysis for activator 1, AD(Red) and oscillatory(Green) region in (a) $\varepsilon - d$ phase space $(k = 7)$. (b) $d - k$ phase space $(\varepsilon = 7)$ (c) $\varepsilon -\ k$ phase space $(d = 5)$.

III. Stability Analysis

I have performed an stability analysis of equation (1) to get the steady state conditions at which those above said two identical FHN system collectively goes to stable equilibrium point. For this, linearizing (1) around the equilibrium point $x^* \equiv (u_i^*, v_i^*, z^*)^T$ by setting $x(t) = x^* + \delta x(t)$ where $u_i^* = -a, v_i^* = \frac{a^3}{3} - a + a\frac{\varepsilon^2}{k}$ for $i = 1{,}2$ I obtain

$$\delta \dot{x} = \frac{1}{r}\begin{pmatrix} \xi & -1 & d & 0 & \varepsilon \\ r & 0 & 0 & 0 & 0 \\ d & 0 & \xi & -1 & \varepsilon \\ 0 & 0 & r & 0 & 0 \\ -\varepsilon\frac{r}{2} & 0 & -\varepsilon\frac{r}{2} & 0 & kr \end{pmatrix}\delta x(t) \qquad (2)$$

where $\xi = 1 - a^2 - d$. From this to get the eigen value the characteristic equation is

$$\begin{vmatrix} \xi - r\lambda & -1 & d & 0 & \varepsilon \\ r & -r\lambda & 0 & 0 & 0 \\ d & 0 & \xi - r\lambda & -1 & \varepsilon \\ 0 & 0 & r & -r\lambda & 0 \\ -\varepsilon\frac{r}{2} & 0 & -\varepsilon\frac{r}{2} & 0 & -kr - r\lambda \end{vmatrix} = 0 \qquad (3)$$

simplifying and solving the (3) one obtains

$$[\lambda d + (1 - \xi\lambda + r\lambda^2)][(k + \lambda)[\lambda d - (1 - \xi\lambda + r\lambda^2)] - \varepsilon^2\lambda] = 0 \qquad (4)$$

so either

$$[\lambda d + (1 - \xi\lambda + r\lambda^2)] = 0 \qquad (5)$$

or

$$[(k + \lambda)[\lambda d - (1 - \xi\lambda + r\lambda^2)] - \varepsilon^2\lambda] = 0 \qquad (6)$$

The steady state of the coupled systems are locally stable if and only if all the roots of the characteristic equations, (5) and (6) are in the open left half complex plane that is real part of eigen value is negative. Applying the Routh-Hurwitz stability criterion I get that the characteristic equations are stable if and only if d ,k and ε satisfy the following inequalities:

$$\begin{gathered} 2d > 1 - a^2 \\ kr > 1 - a^2 \\ [kr - (1 - a^2)][(1 + \varepsilon^2) - k(1 - a^2)] > kr \end{gathered} \qquad (7)$$

From these inequalities (7) one may get the quenching region in different parametric space given by Fig. 2. Red color or deep grey color region indicates the AD region and green or light grey color indicates oscillatory region.

IV. Numerical Results

Numerical simulations are carried out by integrating equation (1) using fourth order Runge-Kutta algorithm with step size $h = 0.001$. Numerical simulation of this system of equations shows that when I coupled both diffusively and indirectly, the system converges to a stable point at the value of $-a$. Here I have used constant Initial value of $u1 = -0.5, v1 = -1, u2 = 0.5, v2 = 0.65, z = 0.3$ and time range $0 - 4000$ with $a = 0.85, r = 0.1$. With suitable value of d, # and k I get quenching region. To start quenching the oscillation, the minimum value of $\varepsilon = 1.59, d = 0.1388$ and $k = 2.739$ is required, i.e. below of those value no suppression of oscillation is possible. The coupled system become stable at $u_i = -a$. As the value of r increased the sytem quenched more rapidly. Here I consider quenching when the amplitude(A) of activators are $A \leq 10^{-4}$.

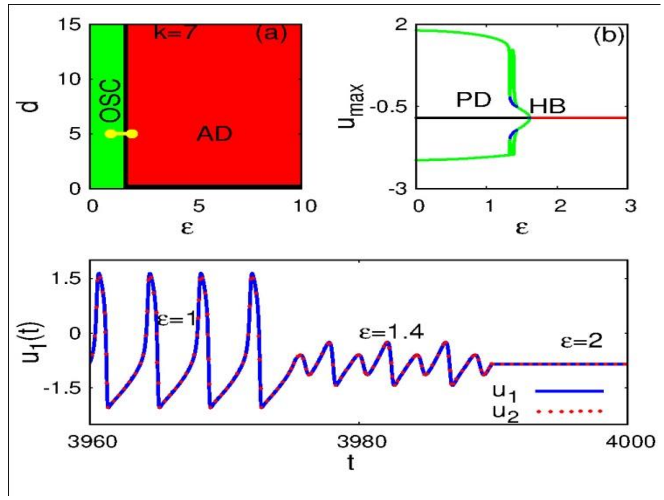

Fig. 3. The two parameter bifurcation diagram in the $\varepsilon - d$ space ($k = 7, r = 0.1, a = 0.85$) (a) Red (dark gray) zone:Amplitude Death and Green (light gray) zone: Oscillation. Solid black line is theoretical curve from stability analysis. (b) Bifurcation diagram with ε, using XPPAUT AUTO 6.1. (c) the time series of activators u i for three different point (1,5), (1.4,5), (2,5) in $\varepsilon - d$ parametric space indcated by yellow dots in (a).Time series for the value $\varepsilon = 1.4$ confirms the period doubling in bifurcation diagram.

A. $\varepsilon - d$ Parametric Space

To study the effect of diffusive coupling on the individual system in presence of environment I first fix the value $k = 7$, keeping $r = 0.1, a = 0.85$ and control the parameter d and ε. The result obtained by numerical simulation is shown by two parameter stability diagram (Fig. 3(a)). I notice that depending on the coupling parameter the system quenched and enters in the AD region through inverse Hopf Bifurcation. In the Fig. 3(a) the amplitude death zone is indicated by red (dark gray) zone and the oscillatory zone by green (light gray). My theoretical analysis matches exactly, indicated by solid black curve. To confirm the results I draw single parameter bifurcation diagram using XPPAUT AUTO 6.1(Fig. 3(b)). The single parameter bifurcation diagram with ε shows that as I increase the value of ε the amplitude of activators eventually seized to the value of $-a$ through a period doubling region in between $1.334 <= \varepsilon <= 1.425$. There occurs an inverse Hopf Bifurcation at $\varepsilon = 1.612$

(here $d = 5$). The time series (Fig. 3(c)) of activators (u_i) is taken on three points along the yellow line indicated in (Fig. 3(a)). It clearly represents the system enters into death state (here $\varepsilon = 2$) through period doubling region (here $\varepsilon = 1.4$) which confirms the bifurcation curve. The time series for $\varepsilon = 1$ and $\varepsilon = 1.4$ also indicate that both the system are in complete synchronisation.

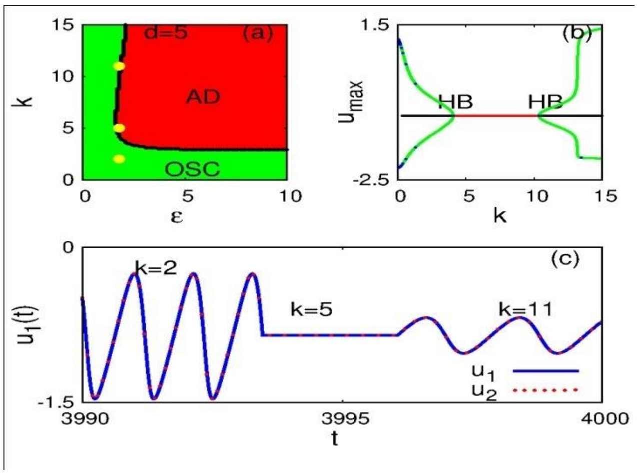

Fig. 4. The two parameter bifurcation diagram in the $\varepsilon - k$ space ($d = 5, r = 0.1, a = 0.85$) (a) Red (dark gray) zone: Amplitude Death and Green (light gray) zone: Oscillation. Solid black line is theoretical curve from stability analysis. (b) Bifurcation diagram with k, using XPPAUT AUTO 6.1. (c) the time series of activators u_i for three different point (1.8,2), (1.8,5), (1.8,11) in $\varepsilon - k$ parametric space indicated by yellow dots in (a).

B. $\varepsilon -$ k Parametric Space

Next I explore the behaviour of coupled system in $\varepsilon - k$ parametric zone keeping the value $d = 5$. The Fig. 4 represents the results. The theoretical results represents by solid black line matches well with the numerical results (Fig. 4(a)). I notice that for $\varepsilon = 1.62$ I get AD within a boundary of k value starting from $k = 5.038$ *to* 7.195 and beyond these value of k I get oscillation. After that value of ε this range of k value is extended in both end and value of k reaches its minimum value 2.739 for getting transition to AD region. This result is interesting in the context of revival of oscillation. For a certain value of ε as k is increased the system first enters into death state and with further increased of k the system revives its oscillatory state. The bifurcation diagram, using XPPAUT AUTO 6.1, (Fig. 4(b)) shows at $k = 4.094$ (here $\varepsilon = 1.8$) the system enters into AD zone(Red/dark gray color) from oscillation(Green /light gray color) through inverse Hopf Bifurcation and again come back to oscillatory state at $k = 10.36$ through Hopf Bifurcation. The time series of activators (u_i) given in Fig. 4(c) shows the transitions between oscillation to AD and vice versa at the points before and after the Hopf points (for Points (1.8,2), (1.8,5) and (1.8,11) indicated by yellow dots in Fig. 4(a)). In this case also both the system are in complete synchronisation.

C. $d - k$ Parametric Space

Lastly I explore the behaviour of coupled system in $d - k$ parametric zone keeping the value $\varepsilon = 7$. The Results are shown in Fig. 5. The solid black line, representation of theoretical analysis, in Fig. 5(a) shows that numerical results agrees with theoretical analysis. The single parameter bifurcation diagram with d, using XPPAUT AUTO 6.1, (Fig. 5(b)) shows at $d = 0.1388$ (here $k = 7$) the system enters into AD region through inverse Hopf Bifurcation. In Fig. 5(c) the time series of activators (u_i) is given for two points represents by yellow dots in Fig. 5(a) which confirms the transition from oscillation (in complete synchronisation) to amplitude death.

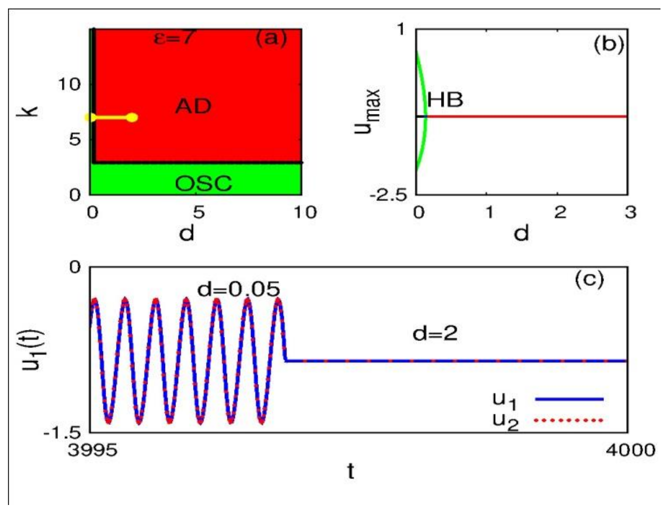

Fig.5. The two parameter bifurcation diagram in the $d - k$ space ($\varepsilon = 7, r = 0.1, a = 0.85$) (a) Red (dark gray) zone: Amplitude Death and Green (light gray) zone: Oscillation. Solid black line is theoretical curve from stability analysis. (b) Bifurcation diagram with k , using XPPAUT AUTO 6.1. (c) the time series of activators u_i for three different point (0.05,7), (2,7) in $d - k$ parametric space indicated by yellow dots in (a).

V. Summary and Conclusions

In this paper I have studied neuronal system model followed by FitzHugh-Nagumo (FHN) system in self sustained periodic firing mode where both diffusive and environment coupling is used. My main objective to observe whether the coupled neural systems may go to amplitude death state with variation of different parameter ε, d *and* k which I explored systematically. I have noticed, using the XPPAUT 6.1 package, that systems enter into AD region from complete synchronized oscillation through inverse Hopf Bifurcation and this is shown in different parametric space. To realize the effect of controlling parameters in transition from oscillation to AD and vice-versa I go through both theoretical analysis and numerical computations. I successfully explore this and represent in different parametric zone and I also see that both theoretical and numerical computations agree well. Presence of environmental coupling helps two diffusively coupled FHN systems to AD state depending on the coupled parameters. Beside this result I also observed that depending on parametric condition or value related to environment the system can also revives it oscillatory state from amplitude death state through Hopf Bifurcation. This result is important in the context of revival of oscillation. As neuronal systems interactions not only depend on their mutual coupling but largely on their environment also, so this study is relevant and have significant result..

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Conflict of Interest

The author declares that he has no conflict of interest.

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