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# Exponentiated Aradhana Distribution with Properties and Applications in Engineering Sciences

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*Abstract-* **In this article, we study some statistical properties of a new distribution namely Exponentiated Aradhana distribution. The Exponentiated Aradhana distribution has two parameters (scale and shape). The different structural properties of the proposed distribution have been obtained. The parameters of the proposed model have been estimated through the maximum likelihood method. Finally, we present a real lifetime data set where it is observed that Exponentiated Aradhana distribution has a better fit compared to two parameter Pranav and two parameter Sujatha distribution.**

*Keywords:* **Exponentiated distribution, Aradhana distribution, Order statistics, Entropies, Reliability analysis, Maximum likelihood Estimation.**

## I. INTRODUCTION

A new concept of distributions was introduced by Gupta et al. (1998) , who discussed a new family of distributions namely the Exponentiated exponential distribution. The family has two parameters scale and shape, which are similar to the weibull or gamma family. Later Gupta and Kundu (2001), studied some properties of the distribution. They observed that many properties of the new family are similar to those of the weibull or gamma family. Hence the distribution can be used an alternative to a weibull or gamma distribution. The two-parameteric gamma and weibull are the most popular distributions for analyzing any lifetime data. The gamma distribution has a lot of applications in different fields other than lifetime distributions. The two parameters of gamma distribution represent the scale and the shape parameter and because of the scale and shape parameter, it has quite a bit of flexibility to analyze any positive real data. But one major disadvantage of the gamma distribution is that, if the shape parameter is not an integer, the distribution function or survival function cannot be expressed in a closed form. This makes gamma distribution little bit unpopular as compared to the Weibull distribution, whose survival function and hazard function are simple and easy to study. Nowadays Exponentiated distributions and their mathematical properties are widely studied for applied science experimental data sets. Pal et al. (2006) studied the Exponentiated weibull family as an extension of weibull distribution. Rodrigues et al. (2017) studied the exponentiated generalized Lindley distribution. Hassan et al. (2017) discussed Exponentiated Lomax geometric distribution with its properties and applications. Nasiru et al., (2018) obtained Exponentiated generalized power series family of distributions. Rather and subramanian (2018) discussed the Exponentiated Mukherjee-Islam distribution which shows more flexibility than the classical distribution. Rather and subramanian (2019) discussed the Exponentiated ishita distribution with properties and Applications. Uwaeme et al. (2019) discussed on the Exponentiated Pranav distribution. Ashour and Eltehiwy (2014) derived and discussed the Exponentiated power lindley distribution. Onyekwere et al. (2021) discussed on the Exponentiated Rama distribution with properties and Applications. Shawky and Zinadah

(2009) obtained the Exponentiated Pareto distribution and discuss its different methods of estimation. Recently, Rather and subramanian (2020), discussed the Exponentiated Garima distribution which shows more flexibility than the classical distribution.

In this paper, we consider a two-parameter Exponentiated Aradhana distribution and study some of its properties. Aradhana distribution is a newly proposed one parameteric distribution formulated by Shanker (2016) for several engineering applications and calculated its various characteristics including stochastic ordering, moments, order statistics, Renyi entropy, stress strength reliability and ML estimation. The two parameters of an Exponentiated Aradhana distribution represent the shape and the scale parameter. It also has the increasing or decreasing failure rate depending of the shape parameter. The density function varies significantly depending of the shape parameter (see Fig.1).

## II. EXPONENTIATED ARADHANA DISTRIBUTION (EAD)

The probability density function of Aradhana distribution is given by

$$g(x) = \frac{\theta^3}{\theta^2 + 2\theta + 2}(1+x)^2 e^{-\theta x}; \; x > 0, \theta > 0 \tag{1}$$

and the cumulative distribution function of Aradhana distribution is given by

$$G(x) = 1 - \left(1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x}; \; x > 0, \theta > 0 \tag{2}$$

A random variable $X$ is said to have an Exponentiated distribution, if its cumulative distribution function is given by

$$F_\alpha(x) = \left(G(x)\right)^\alpha \;; \; x \in R', \; \alpha > 0 \tag{3}$$

Then $X$ is said to have an Exponentiated distribution.

The probability density function of X is given by

$$f_\alpha(x) = \alpha\left(G(x)\right)^{\alpha-1} g(x) \tag{4}$$

By Substituting (2) in (3), we will obtain the cumulative distribution function of Exponentiated Aradhana distribution

$$F_\alpha(x) = \left(1 - \left(1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x}\right)^\alpha$$

$$; \; x > 0, \theta > 0, \alpha > 0 \tag{5}$$

and the probability density function of Exponentiated Aradhana distribution can be obtained as

$$f_\alpha(x) = \frac{\alpha\theta^3(1+x)^2 e^{-\theta x}}{\theta^2 + 2\theta + 2}\left(1 - \left(1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x}\right)^{\alpha-1}$$

$$; x > 0, \theta > 0, \alpha > 0 \tag{6}$$

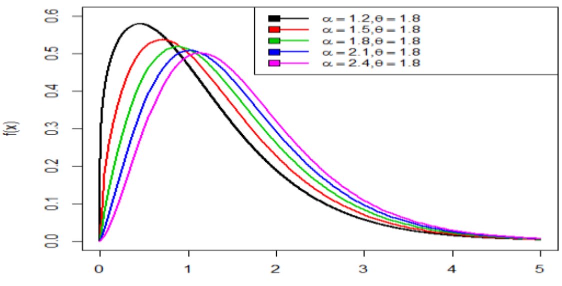

Fig.1 Pdf plot of Exponentiated Aradhana distribution

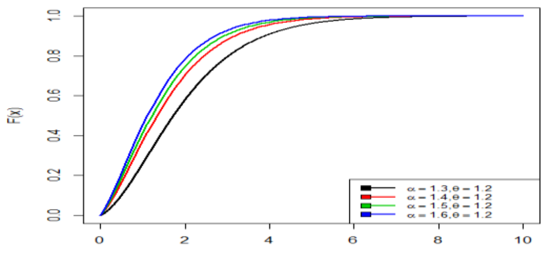

Fig.2 Cdf plot of Exponentiated Aradhana distribution

## III. RELIABILITY MEASURES

In this section, we will obtain the survival function, hazard function and Reverse hazard rate function of the Exponentiated Aradhana distribution.

The survival function of Exponentiated Aradhana distribution is given by

$$S(x) = 1 - \left(1 - \left(1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta x}\right)^\alpha$$

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality and is given by

$$h(x)=\left(\frac{\dfrac{\alpha\theta^3(1+x)^2e^{-\theta x}}{\theta^2+2\theta+2}\times\left(1-\left(1+\dfrac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}}{1-\left(1-\left(1+\dfrac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha}}\right)$$

The reverse hazard rate of Exponentiated Aradhana distribution is given by

$$h_r(x)=\frac{\alpha\theta^3(1+x)^2e^{-\theta x}}{\theta x(\theta x+2\theta+2)e^{-\theta x}}$$

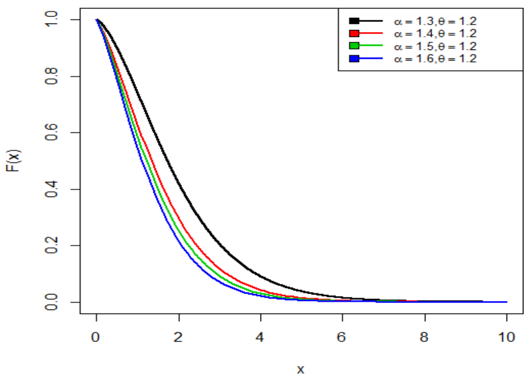


Fig.3 Survival plot of Exponentiated Aradhana distribution

## IV. STATISTICAL PROPERTIES

In this section, we will define and discuss the different statistical properties of the proposed Exponentiated Aradhana distribution.

### *A. Moments*

Suppose X is a random variable following Exponentiated Aradhana distribution with parameters $\alpha$ and $\theta$, then the rth order moment $E(X^r)$ for a given probability distribution is given by

$$\mu_r' = E(X^r)=\int_0^\infty x^r f_\alpha(x)dx$$

$$=\int_0^\infty x^r\frac{\alpha\theta^3(1+x)^2e^{-\theta x}}{\theta^2+2\theta+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}dx$$

$$=\frac{\alpha\theta^3}{\theta^2+2\theta+2}\int_0^\infty x^r(1+x)^2e^{-\theta x}$$

$$\times\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}dx \qquad (7)$$

Using Binomial expansion of

$$\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}$$

$$=\sum_{i=0}^{\infty}\binom{\alpha-1}{i}\left\{\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right\}^i(-1)^i$$

Equation (7) will become

$$=\frac{\alpha\theta^3}{\theta^2+2\theta+2}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}$$

$$\times\int_0^\infty x^r(1+x)^2e^{-\theta x(1+i)}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)^i dx$$

Again

(8)

using Binomial expansion of

$$\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)^i$$

$$=\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)^k$$

Equation (8) becomes

$$\mu_r' = E(X^r) = \frac{\alpha\theta^3}{\theta^2 + 2\theta + 2} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \binom{i}{k}$$

$$\times \left( \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right)^k \int_o^{\infty} x^r (1+x)^2 e^{-\theta x(1+i)} dx$$

After simplification, we obtain

$$\mu_r' = E(X^r) = \alpha\theta^3 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \binom{i}{k}$$

$$\times \frac{(\theta^2 + 2\theta^2 + 2\theta)^k}{(\theta^2 + 2\theta + 2)^{k+1}}$$

$$\times \left( \frac{\theta(1+i)^2 \Gamma(r+4k+1) + \Gamma(r+4k+3) + 2\theta(1+i)\Gamma(r+4k+2)}{\theta(1+i)^{r+4k+3}} \right) \tag{9}$$

Since equation (9) is a convergent series for all r ≥ 0, therefore all the moments exist.

Therefore

$$\mu_1' = E(X) = \alpha\theta^3 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \binom{i}{k} \frac{(\theta^2 + 2\theta^2 + 2\theta)^k}{(\theta^2 + 2\theta + 2)^{k+1}}$$

$$\times \left( \frac{\theta(1+i)^2 \Gamma(4k+2) + \Gamma(4k+4) + 2\theta(1+i)\Gamma(4k+3)}{\theta(1+i)^{4k+4}} \right)$$

$$\mu_2' = E(X^2) = \alpha\theta^3 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \binom{i}{k} \frac{(\theta^2 + 2\theta^2 + 2\theta)^k}{(\theta^2 + 2\theta + 2)^{k+1}}$$

$$\times \left( \frac{\theta(1+i)^2 \Gamma(4k+3) + \Gamma(4k+5) + 2\theta(1+i)\Gamma(4k+4)}{\theta(1+i)^{4k+5}} \right)$$

Therefore the Variance of X can be obtained as

$$V(X) = E(X^2) - (E(X))^2$$

*B. Harmonic mean*

The Harmonic mean for the proposed Exponentiated Aradhana distribution can be obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^{\infty} \frac{1}{x} f_{\alpha}(x) dx$$

$$= \int_0^{\infty} \frac{1}{x} \frac{\alpha\theta^3 (1+x)^2 e^{-\theta x}}{\theta^2 + 2\theta + 2}$$

$$\times \left( 1 - \left( 1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right) e^{-\theta x} \right)^{\alpha-1} dx$$

$$= \frac{\alpha\theta^3}{\theta^2 + 2\theta + 2} \int_0^{\infty} \frac{1}{x} (1+x)^2 e^{-\theta x}$$

$$\times \left( 1 - \left( 1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right) e^{-\theta x} \right)^{\alpha-1} dx \tag{10}$$

Using Binomial expansion in equation (10), we get

$$= \frac{\alpha\theta^3}{\theta^2 + 2\theta + 2} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \int_0^{\infty} \frac{1}{x} (1+x)^2 e^{-\theta x(1+i)}$$

$$\times \left( 1 + \frac{\theta x(\theta x + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right)^i dx \tag{11}$$

On using Binomial expansion in equation (11), we obtain

$$H.M = \frac{\alpha\theta^3}{\theta^2 + 2\theta + 2} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \binom{i}{k}$$

$$\times \left( \frac{\theta^2 x^2 + 2\theta^2 x + 2\theta x)}{\theta^2 + 2\theta + 2} \right)^k \int_0^{\infty} \frac{1}{x} (1+x)^2 e^{-\theta x(1+i)} dx \tag{12}$$

After the simplification of equation (12), we obtain

$$H.M=\alpha\theta^3\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{\left(\theta^2+2\theta^2+2\theta\right)^k}{\left(\theta^2+2\theta+2\right)^{k+1}}$$

$$\times\left(\frac{\theta(1+i)\Gamma(4k+1)+2\theta(1+i)\Gamma(4k+1)+\Gamma(4k+2)}{\theta(1+i)^{4k+2}}\right)$$

*C. Moment Generating Function and Characteristics Function*

In probability theory and statistics, the moment generating function of a real valued random variable is an alternative specification of its probability distribution. As its name implies the moment generating function can be used to compute the moments of a distribution. Let X have an Exponentiated Aradhana distribution, then the moment generating function of X is obtained as

$$M_X(t)=E\left(e^{tx}\right)=\int_0^{\infty}e^{tx}f_{\alpha}(x)dx$$

Using Taylor's series, we get

$$M_X(t)=\int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+....\right)f_{\alpha}(x)dx$$

$$=\int_0^{\infty}\sum_{j=0}^{\infty}\frac{t^j}{j!}x^j f_{\alpha}(x)dx$$

$$=\sum_{j=0}^{\infty}\frac{t^j}{j!}\mu_j{}'$$

$$M_X(t)=\alpha\theta^3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{t^j}{j!}\frac{\left(\theta^2+2\theta^2+2\theta\right)^k}{(\theta^2+2\theta+2)^{k+1}}$$

$$\times\left(\frac{\theta(1+i)^2\Gamma(j+4k+1)+\Gamma(j+4k+3)+2\theta(1+i)\Gamma(j+4k+2)}{\theta(1+i)^{j+4k+3}}\right)$$

Similarly, the characteristic function of Exponentiated Aradhana distribution is given by

$$\varphi_X(t)=\alpha\theta^3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{mt^j}{j!}$$

$$\times\frac{\left(\theta^2+2\theta^2+2\theta\right)^k}{(\theta^2+2\theta+2)^{k+1}}$$

$$\times\left(\frac{\theta(1+i)^2\Gamma(j+4k+1)+\Gamma(j+4k+3)+2\theta(1+i)\Gamma(j+4k+2)}{\theta(1+i)^{j+4k+3}}\right)$$

## V. ORDER STATISTICS

Order statistics represents the arranging of samples in an ascending order. Order statistics also has wide field in reliability and life testing. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots X_n$ drawn from the continuous population with probability density function $f_x(x)$ and cumulative distribution function $F_x(x)$, then the pdf of $r^{th}$ order statistics $X_{(r)}$ can be written as

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r} \qquad (13)$$

Substitute the values of equation (5) and (6) in equation (13), we will obtain the pdf of $r^{th}$ order statistics $X_{(r)}$ for Exponentiated Aradhana distribution and is given by

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{\alpha\theta^3(1+x)^2e^{-\theta x}}{\theta^2+2\theta+2}$$

$$\times\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}$$

$$\times\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha(r-1)}$$

$$\times\left(1-\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha}\right)^{n-r}$$

Therefore the probability density function of higher order statistics $X_{(n)}$ for Exponentiated Aradhana distribution can be obtained as

$$f_{x(n)}(x) = n\frac{\alpha\theta^3(1+x)^2 e^{-\theta x}}{\theta^2+2\theta+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1} \times\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha(n-1)}$$

and the pdf of first order statistics $X_{(1)}$ for Exponentiated Aradhana distribution can be obtained as

$$f_{x(1)}(x) = n\frac{\alpha\theta^3\left(1+x\right)^2 e^{-\theta x}}{\theta^2+2\theta+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1} \times\left(1-\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha}\right)^{n-1}$$

## VI. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we will discuss the maximum likelihood estimation for estimating the parameters of Exponentiated Aradhana distribution. Let $X_1, X_2, \ldots X_n$ be the random sample of size $n$ from the Exponentiated Aradhana distribution, then the likelihood function can be written as

$$L(\alpha,\theta) = \frac{\left(\alpha\theta^3\right)^n}{(\theta^2+2\theta+2)^n} \times\prod_{i=1}^{n}\left((1+x)^2 e^{-\theta x}\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}\right)$$

The log likelihood function is given by

$$\log L(\alpha,\theta) = n\log\alpha + 3n\log\theta - n\log(\theta^2+2\theta+2) + 2\sum_{i=1}^{n}\log(1+x) - \theta\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n}\log\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right) \quad (14)$$

The maximum likelihood estimates of $\alpha$, $\theta$ which maximizes (14), must satisfy the following equations given by

$$\frac{\partial\log L}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right) = 0$$

$$\Rightarrow \hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\log\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)}$$

$$\frac{\partial\log L}{\partial\theta} = \frac{3n}{\theta} - n\left(\frac{(2\theta+2)}{\theta^2+2\theta+2}\right) - \sum_{i=1}^{n}x_i + (\alpha-1)\psi\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right) = 0$$

Where ψ (.) is the digamma function.

Hence, it is very difficult to estimate the value of $\theta$ because the above likelihood equation is too complicated. Therefore we use R and wolfram mathematics for estimating the required parameter $\theta$.

## VII. INFORMATION MEASURES OF EXPONENTIATED ARADHANA DISTRIBUTION

### *A.Renyi Entropy*

The Renyi entropy was given by Alfred Renyi (1961) in the context of fractual dimension estimation, the Renyi entropy forms the basis of the concept of generalized dimensions. The Renyi entropy is important in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. Entropies quantify the diversity, uncertainty, or randomness of a system. For a given probability distribution, Renyi entropy is given by

$$e(\beta) = \frac{1}{1-\beta}\log\left(\int_0^\infty f^\beta(x)dx\right)$$

Where, $\beta > 0$ and $\beta \neq 1$

$$= \frac{1}{1-\beta}\log\left(\int_0^\infty\left\{\begin{array}{l}\frac{\alpha\theta^3(1+x)^2 e^{-\theta x}}{\theta^2+2\theta+2} \\ \times\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}\end{array}\right\}^\beta dx\right)$$

$$= \frac{1}{1-\beta}\log\left(\begin{array}{l}\left(\frac{\alpha\theta^3}{\theta^2+2\theta+2}\right)^\beta \int(1+x)^{2\beta}e^{-\theta\beta x} \\ \left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta x}\right)^{\beta(\alpha-1)} dx\end{array}\right)$$

(15)

Using binomial expansion in (15), we get

$$= \frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^{3}}{\theta^{2}+2\theta+2}\right)^{\beta}\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta(\alpha-1)}{i}\int_{0}^{\infty}(1+x)^{2\beta}e^{-\theta x(\beta+i)}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^{2}+2\theta+2}\right)^{i}dx\right) \tag{16}$$

Again using binomial expansion in (16), we get

$$= \frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^{3}}{\theta^{2}+2\theta+2}\right)^{\beta}\times\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i}\binom{\beta(\alpha-1)}{i}\binom{i}{k}\times\left(\frac{\theta^{2}x^{2}+2\theta^{2}x+2\theta x}{\theta^{2}+2\theta+2}\right)^{k}\times\int_{0}^{\infty}(1+x)^{2\beta}e^{-\theta x(\beta+i)}dx\right) \tag{17}$$

After the simplification of (17) we obtain

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\alpha\theta^{3}\right)^{\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i}\times\binom{\beta(\alpha-1)}{i}\binom{i}{k}\binom{2\beta}{j}\times\frac{\left(\theta^{2}+2\theta^{2}+2\theta\right)^{k}}{\left(\theta^{2}+2\theta+2\right)^{\beta+k}}\frac{\Gamma(4k+j+1)}{\theta(\beta+i)^{4k+j+1}}\right)$$

*B. Tsallis Entropy*

The concept of Tsallis entropy was introduced in 1988 by Constantino Tsallis. A generalization of Boltzmann-Gibbs (B-G) statistical mechanics initiated by Tsallis has gained a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy for a continuous random variable it is defined as

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\int_{0}^{\infty}f^{\lambda}(x)dx\right)$$

$$=\frac{1}{\lambda-1}\left(1-\int_{0}^{\infty}\left\{\frac{\alpha\theta^{3}(1+x)^{2}e^{-\theta x}}{\theta^{2}+2\theta+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^{2}+2\theta+2}\right)e^{-\theta x}\right)^{\alpha-1}\right\}^{\lambda}dx\right)$$

$$=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^{3}}{\theta^{2}+2\theta+2}\right)^{\lambda}\int_{0}^{\infty}(1+x)^{2\lambda}e^{-\lambda\theta x}\left(1-\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^{2}+2\theta+2}\right)e^{-\theta x}\right)^{\lambda(\alpha-1)}dx\right) \tag{18}$$

Using binomial expansion in (18), we get

$$=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^{3}}{\theta^{2}+2\theta+2}\right)^{\lambda}\sum_{i=0}^{\infty}(-1)^{i}\binom{\lambda(\alpha-1)}{i}\int_{0}^{\infty}(1+x)^{2\lambda}e^{-\theta x(\lambda+i)}\left(1+\frac{\theta x(\theta x+2\theta+2)}{\theta^{2}+2\theta+2}\right)^{i}dx\right) \tag{19}$$

Again using binomial expansion in (19), we obtain

$$=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^{3}}{\theta^{2}+2\theta+2}\right)^{\lambda}\times\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i}\binom{\lambda(\alpha-1)}{i}\binom{i}{k}\left(\frac{\theta^{2}x^{2}+2\theta^{2}x+2\theta x}{\theta^{2}+2\theta+2}\right)^{k}\times\int_{0}^{\infty}(1+x)^{2\lambda}e^{-\theta x(\lambda+i)}dx\right) \tag{20}$$

After the simplification of (20), we get

$$S_{\lambda} = \frac{1}{\lambda - 1}\left( \begin{array}{l} 1 - \left(\alpha\theta^{3}\right)^{\lambda} \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{\infty} \sum\limits_{j=0}^{\infty} (-1)^{i} \binom{\lambda(\alpha - 1)}{i} \\ \times \binom{i}{k}\binom{2\lambda}{j} \dfrac{\left(\theta^{2} + 2\theta^{2} + 2\theta\right)^{k}}{\left(\theta^{2} + 2\theta + 2\right)^{\lambda + k}} \\ \times \dfrac{\Gamma(4k + j + 1)}{\theta(\lambda + i)^{4k + j + 1}} \end{array} \right)$$

## VIII. DATA ANALYSIS

In this section, we use the two real-life data sets in Exponentiated Aradhana distribution and the model has been compared with two parameter Pranav and two parameter Sujatha distributions.

The first data set represents the breaking stress of carbon fibres of 50 mm length (GPa) reported by Nicholas and Padgett (2006) and the data set is presented below in table 1

| Data set 1 | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.39 | 0.85 | 1.08 | 1.25 | 1.47 | 1.57 | 1.61 | 1.61 |
| 1.69 | 1.80 | 1.84 | 1.87 | 1.89 | 2.03 | 2.03 | 2.05 |
| 2.12 | 2.35 | 2.41 | 2.43 | 2.48 | 2.50 | 2.53 | 2.55 |
| 2.55 | 2.56 | 2.59 | 2.67 | 2.73 | 2.74 | 2.79 | 2.81 |
| 2.82 | 2.85 | 2.87 | 2.88 | 2.93 | 2.95 | 2.96 | 2.97 |
| 3.09 | 3.11 | 3.11 | 3.15 | 3.15 | 3.19 | 3.22 | 3.22 |
| 3.27 | 3.28 | 3.31 | 3.31 | 3.33 | 3.39 | 3.39 | 3.56 |
| 3.60 | 3.65 | 3.68 | 3.70 | 3.75 | 4.20 | 4.38 | 4.42 |
| 4.70 | 4.90 | | | | | | |

The second data set represent the strength data of glass of the aircraft window reported by Fuller et al (1994).The data set is provided below in table 2.

| Data set 2 | | | | | | |
|---|---|---|---|---|---|---|
| 18.83 | 20.80 | 21.657 | 23.03 | 23.23 | 24.05 | 24.321 |
| 25.50 | 25.52 | 25.80 | 26.69 | 26.77 | 26.78 | 27.05 |
| 27.67 | 29.90 | 31.11 | 33.20 | 33.73 | 33.76 | 33.89 |
| 34.76 | 35.75 | 35.91 | 36.98 | 37.08 | 37.09 | 39.58 |
| 44.045 | 45.29 | 45.381 | | | | |

In order to compare the Exponentiated Aradhana distribution with two parameter Pranav and two parameter Sujatha distributions. we consider the Criterion values like BIC (Bayesian information criterion), AIC (Akaike information criterion), AICC (Corrected Akaike information criterion) and -2logL. The better distribution corresponds to lesser values of AIC, BIC, AICC and -2logL. For calculating the criterion values like AIC, BIC, AICC and -2logL can be evaluated by using the formulas as follows.

$$AIC = 2k - 2\log L, \quad AICC = AIC + \frac{2k(k+1)}{n-k-1} \quad \text{and}$$

$$BIC = k\log n - 2\log L$$

Where $k$ is the number of parameters in the statistical model, $n$ is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model.

Table 3 shows comparison of Exponentiated Aradhana distribution with two parameter Pranav and two parameter Sujatha distribution

| Data sets | Distribution | MLE | S.E | -2logL | AIC | BIC | AICC |
|---|---|---|---|---|---|---|---|
| | Exponentiated Aradhana | $\hat{\alpha} = 5.5104$<br>$\hat{\theta} = 1.4808$ | $\hat{\alpha} = 1.3217$<br>$\hat{\theta} = 0.1180$ | 185.1499 | 189.1499 | 193.5293 | 189.3403 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | Two Parameter Pranav | $\hat{\alpha} = 0.0010$ $\hat{\theta} = 1.4486$ | $\hat{\alpha} = 0.0473$ $\hat{\theta} = 0.0982$ | 193.5953 | 197.5953 | 201.9746 | 197.7857 |
| | Two Parameter Sujatha | $\hat{\alpha} = 0.0010$ $\hat{\theta} = 0.9686$ | $\hat{\alpha} = 0.0054$ $\hat{\theta} = 0.0312$ | 213.4739 | 217.4739 | 221.8532 | 217.6643 |
| 2 | Exponentiated Aradhana | $\hat{\alpha} = 19.1870$ $\hat{\theta} = 0.2200$ | $\hat{\alpha} = 10.3667$ $\hat{\theta} = 0.0265$ | 208.1654 | 212.1654 | 215.0333 | 212.5939 |
| | Two Parameter Pranav | $\hat{\alpha} = 0.1035$ $\hat{\theta} = 0.1298$ | $\hat{\alpha} = 0.9854$ $\hat{\theta} = 0.0051$ | 232.7729 | 236.7729 | 239.6409 | 237.2014 |
| | Two Parameter Sujatha | $\hat{\alpha} = 0.0010$ $\hat{\theta} = 0.0958$ | $\hat{\alpha} = 0.6755$ $\hat{\theta} = 0.0064$ | 241.3064 | 245.3064 | 248.1744 | 245.7349 |

From table 3, it can be observed that the Exponentiated Aradhana distribution have the lesser AIC, BIC, AICC and -2log$L$ values as compared to the two parameter Pranav and two parameter Sujatha distributions. Hence we can conclude that the Exponentiated Aradhana distribution leads to a better fit than the two parameter Pranav and two parameter Sujatha distributions.

## CONCLUSION

In the present manuscript, we have introduced a new model of the Aradhana distribution called as Exponentiated Aradhana distribution with two parameters (scale and shape). The subject distribution is generated by using the Exponentiated technique and the parameters have been obtained by using the maximum likelihood estimator. Some statistical properties along with reliability measures are discussed. The new distribution with its applications in real life-time data has been demonstrated. Finally, the results of two real lifetime data sets have been compared over two parameter Pranav and two parameter Sujatha distributions and it has been found that the Exponentiated Aradhana distribution provides better fit than the two parameter Pranav and two parameter Sujatha distribution.

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