Volume 66, Issue 1, 2022

Journal of Scientific Research

of

The Banaras Hindu University

Factorial Tripling Formula Using Arithmetic and Geometric Means and Approximation of Factorials

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Abstract: **Purpose of writing this paper is to introduce a formula to approximate the value of factorial of an integer greater than one by use of arithmetic and geometric means of three consecutive integers. Methodology applied is the use of approximation of arithmetic mean (AM) to geometric mean (GM) of three closely placed large positive integers in arithmetic progression. The approximation is further improved by applying a correction factor. Three consecutive multiplying integers of a factorial of an integer are grouped and the bunches so formed, are replaced with the cube of their respective arithmetic mean, thus reducing the number of terms to one third. Bunching successively reduces the terms finally to one. Multiplying the final term with cumulative correction factor, yields the result. The method is simple, unattempted, unique, innovative and yields precise results.**

Index Terms: **Approximation, Arithmetic Mean, Bunching, Correction Multiplier, Factorial, Geometric Mean, Telescopic. Geometric Series.**

I. Introduction

Factorial of an integer $n > 0$, has n multiplying terms $1, 2, 3, \dots, n$ and it is given by equation

$$n! = 1 \cdot 2 \cdot 3 \dots (n-2) \cdot (n-1) \cdot n. \quad (1)$$

If integer $(n-1)$ is divisible by three and first multiplying term of factorial of a positive integer being 1, is excluded, remaining $(n-1)$ terms can be grouped in bunches of three terms, forming $(n-1)/3$ bunches. Since consecutive multiplying terms of a factorial of a positive integer, are in arithmetic progression, therefore, GM of three consecutive terms of the bunch can be approximated to its AM after multiplication with correction factor. Each bunch, then can be replaced with cube of respective AM of the terms of the bunch and the product of the cube of respective AM, when multiplied with correction factor, yields value of the factorial of the corresponding integer. At some places in this paper, correction factor is also written as correction multiplier on account of the fact that it multiplies with product of cubes of AM's to achieve correction. In fact, both connote to same meaning.

Lemma 1: *Product of three consecutive terms $(a-1)(a)(a+1)$ of a bunch can be approximated to a^3, if a is large. If a is not large, then product of the terms $(a-1)(a)(a+1)$ of a bunch can be approximated to a^3 multiplied with correction factor c, where $c = (1 - 1/a^2)$.*

Proof: Admittedly multiplying terms $1, 2, 3, \dots, (n-2), (n-1), n$, are in arithmetic progression with a common difference of 1. If we take three consecutive terms say $(a-1), (a), (a+1)$, their AM is 'a' and GM is $(a^3 - a)^{1/3}$. Obviously, $AM > GM$ as $a > (a^3 - a)^{1/3}$. Ratio of AM and GM is $a/(a^3 - a)^{1/3}$ which is always more than 1. If $a \gg 1$, then this ratio approximates to 1 or $AM/GM \simeq 1$. That means $(a^3 - a)$ is replaceable with a^3, when a is large. If a is not large, then for replacing $(a^3 - a)$ with a^3, a correction factor of $(1 - 1/a^2)$, will have to be multiplied with a^3.

Example: Let a be large integer equal to 10000, then $(a-1)(a)(a+1) = 9.9999999 \times 10^{11}$ and $a^3 = 10^{12}$. Therefore, 9.9999999×10^{11} can be approximated to 10^{12} with in percentage error of $1.00000001 \times 10^{-6}$ which is negligible. If a is small integer say 3, then $(a-1)(a)(a+1) = 24$ and $a^3 = 27$. 27 can not be approximated to 24 as there is an appreciable percentage error of 8.3333333 and needs

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DOI: 10.37398/JSR.2022.660141

multiplication with correction factor, $c = (1 - 1/3^2)$ or $c = 0.8888888889$.

If integer $(n-1)$ is divisible by 3, terms of $n!$, given by equation (1), can be grouped in $(n-1)/3$ bunches, where one bunch comprises of three consecutive terms. In that case, equation (1) can be written with bunches of three terms as given below.

$$n! = 1 \cdot \{2 \cdot 3 \cdot 4\} \cdot \{5 \cdot 6 \cdot 7\} \dots \{(n-2) \cdot (n-1) \cdot n\}. \quad (2)$$

On replacing each bunch with cube of its AM, multiplied with respective correction factor, we get

$$n! = 1 \cdot \left\{3^3\left(1 - \tfrac{1}{3^2}\right)\right\}\left\{6^3\left(1 - \tfrac{1}{6^2}\right)\right\}\left\{9^3\left(1 - \tfrac{1}{9^2}\right)\right\} \dots [\{1 - 1/(n-1)^2\}(n-1)^2]\,. \quad (3)$$

Lemma 2: *Factorial n can be shrunk to factorial* $(n-1)/3$ *where integer* $n-1$ *is divisible by* 3. *by forming bunches of three consecutive terms excluding first term one. The value of n!, after bunching, will equal* $\{C_{(n-1)/3}\} \cdot (3^{n-1}) \cdot \left\{\left(\frac{n-1}{3}\right)!\right\}^3$, *where* $C_{(n-1)/3}$ *is a correction factor for* $(n-1)/3$ *bunches and equals* $(1 - 1/3^2)(1 - 1/6^2)(1 - 1/9^2) \dots \{1 - 1/(n-1)^2\}$.

Proof: Equation (3), after bunching can also be written as

$$n! = \left(C_{\frac{n-1}{3}}\right) \cdot \{3^3 \cdot 6^3 \cdot 9^3 \dots (n-7)^3 \cdot (n-4)^3 \cdot (n-1)^3\}$$

where $C_{(n-1)/3}$ is a correction multiplier for $(n-1)/3$ bunches and is given by equation

$$C_{(n-1)/3} = \left[\left(1 - \tfrac{1}{3^2}\right) \cdot \left(1 - \tfrac{1}{6^2}\right) \cdot \left(1 - \tfrac{1}{9^2}\right) \dots \left\{1 - \tfrac{1}{(n-7)^2}\right\} \cdot \left\{1 - \tfrac{1}{(n-4)^2}\right\}\left\{1 - \tfrac{1}{(n-1)^2}\right\}\right]. \quad (4)$$

Or

$$n! = \{C_{(n-1)/3}\}(3^{n-1})\left\{1 \cdot 2 \cdot 3. \dots \left(\tfrac{n-7}{3}\right) \cdot \left(\tfrac{n-4}{3}\right) \cdot \left(\tfrac{n-1}{3}\right)\right\}^3$$

$$= \{C_{(n-1)/3}\} \cdot (3^{n-1}) \cdot \left\{\left(\tfrac{n-1}{3}\right)!\right\}^3 \quad (5)$$

Example: Let $n = 13$, then according to equation (5), $n! = (C_4) \cdot (3^{12}) \cdot \{4!\}^3$, where $C_4 = [(1 - 1/3^2)(1 - 1/6^2)(1 - 1/9^2)\{1 - 1/12^2\}]$, according to equation (4). On calculation, $C_4 = 0.8476011448$. Therefore, $n! = (.8476011448) \cdot (3^{12}) \cdot \{4!\}^3 = 6.2270208 \times 10^9$. Actual value of 13! is 6.2270208×10^9 and thus both are equal

II. Correction Multiplier

A. Derivation of general function for correction multiplier C_x

Equation (4) can be written as

$$C_{\frac{n-1}{3}} = \left(1 - \tfrac{1}{9 \cdot 1^2}\right)\left(1 - \tfrac{1}{9 \cdot 2^2}\right)\left(1 - \tfrac{1}{9 \cdot 3^2}\right) \dots \left\{1 - \tfrac{1}{9 \cdot \left(\frac{n-1}{3}\right)^2}\right\} \quad (6)$$

Writing $(n-1)/3$ as x, the equation takes the form

$$C_x = \left(1 - \tfrac{1}{9 \cdot 1^2}\right)\left(1 - \tfrac{1}{9 \cdot 2^2}\right)\left(1 - \tfrac{1}{9 \cdot 3^2}\right) \dots \left\{1 - \tfrac{1}{9 \cdot x^2}\right\}. \quad (7)$$

Or

$$C_x = \prod_{x=1}^{(n-1)/3} \{1 - 1/(9x^2)\}.$$

Or

$$C_x = C_{x-1} \cdot \{1 - 1/(n-1)^2\},$$

where

$$C_{x-1} = \prod_{x=1}^{(n-4)/3} \{1 - 1/(9x^2)\}.$$

Symbol $\prod_{x=1}^{(n-1)/3}\{1 - 1/(9x^2)\}$ denotes product of terms $\{1 - 1/(9x^2)\}$, when x varies from bunch 1 to bunch $(n-1)/3$. Taking logarithm to the base of natural number e,

$$ln\ C_x = ln\left(1 - \frac{1}{9 \cdot 1^2}\right) + ln\left(1 - \frac{1}{9 \cdot 2^2}\right) + ln\left(1 - \frac{1}{9 \cdot 3^2}\right) + \dots + ln\left(1 - \frac{1}{9 \cdot x^2}\right).$$

This equation can also be written in the form

$$ln\, C_x = \sum_{x=1}^{(n-1)/3} ln\{1 - 1/(9x^2)\}$$

and $\sum_{x=1}^{(n-1)/3} ln\{1 - 1/(9x^2)\}$ can be approximated to $ln \int ln\{1 - 1/(9x^2\}\, dx$. Therefore,

$$ln\, C_x \simeq \int \{1 - (1/9x^2)\}\, dx, \quad (8)$$

where $\sum_{x=1}^{(n-1)/3} ln\{1 - 1/(9x^2)\}$ denotes sum of terms $ln\{1 - 1/(9x^2)$, when x varies from 1 to $(n-1)/3$; $\int ln\{1 - 1/(9x^2)\}\, dx$ denotes integration of $ln\{1 - 1/(9x^2)\}$ with respect x; $ln\, C_x$ denotes natural logarithm of correction factor for x bunches and symbol $\simeq$ is a sign of approximation. It is submitted that since the method used here for calculating value of n factorial and discovering *Factorial Tripling Formula*, are based on approximation and also the fact that approximation of value of n factorial, invokes exponential terms, sign of approximation ($\simeq$) will be used in this paper in stead of sign of equality ($=$).

Assuming number of bunches x to be large, will mean value of term $1/(9x^2)$ is small and, then $ln\{1 - 1/(9x^2)\}$ can be expanded,

$$ln\left(1 - \frac{1}{9x^2}\right) \simeq -\left(\frac{1}{9x^2} + \frac{1}{162x^4} + \frac{1}{2187x^6} + \dots up\ to\ \infty\right).$$

On integrating right hand side (RHS) with respect to x, the equation takes the form,

$$\int \ln\left(1 - \tfrac{1}{9x^2}\right) dx \simeq \left(\tfrac{1}{9x} + \tfrac{1}{486x^3} + \tfrac{1}{10935x^5} + \dots\ up\ to\ \infty\right) + b,$$

where b is constant of integration. At $x = 1$, logarithmic correction multiplier is $ln\,(8/9)$. Therefore,

$$ln\frac{8}{9} - \left(\frac{1}{9x} + \frac{1}{486x^3} + \frac{1}{10935x^5} + \dots\ up\ to\ \infty\right) \simeq b.$$

On putting this value of b in above equation and rearranging,

$$\int ln\left(1 - \tfrac{1}{9x^2}\right) dx \simeq ln\tfrac{8}{9} + \tfrac{1}{9}\left(\tfrac{1}{x} - 1\right) + \tfrac{1}{486}\left(\tfrac{1}{x^3} - 1\right) + \tfrac{1}{10935}\left(\tfrac{1}{x^5} - 1\right). \quad (9)$$

Taking antilog,

$$C_x = \tfrac{8}{9}.exp\left\{\tfrac{1}{9}\left(\tfrac{1}{x} - 1\right) + \tfrac{1}{486}\left(\tfrac{1}{x^3} - 1\right) + \tfrac{1}{10935}\left(\tfrac{1}{x^5} - 1\right)\right\}, \quad (10)$$

where $exp\left\{\tfrac{1}{9}\left(\tfrac{1}{x} - 1\right) + \tfrac{1}{486}\left(\tfrac{1}{x^3} - 1\right) + \tfrac{1}{10935}\left(\tfrac{1}{x^5} - 1\right)\right\}$ is written for exponential term $e^{\left\{\frac{1}{9}\left(\frac{1}{x}-1\right)+\frac{1}{486}\left(\frac{1}{x^3}-1\right)+\frac{1}{10935}\left(\frac{1}{x^5}-1\right)\right\}}$.

B. Correction of smooth curve to match stair steps figure

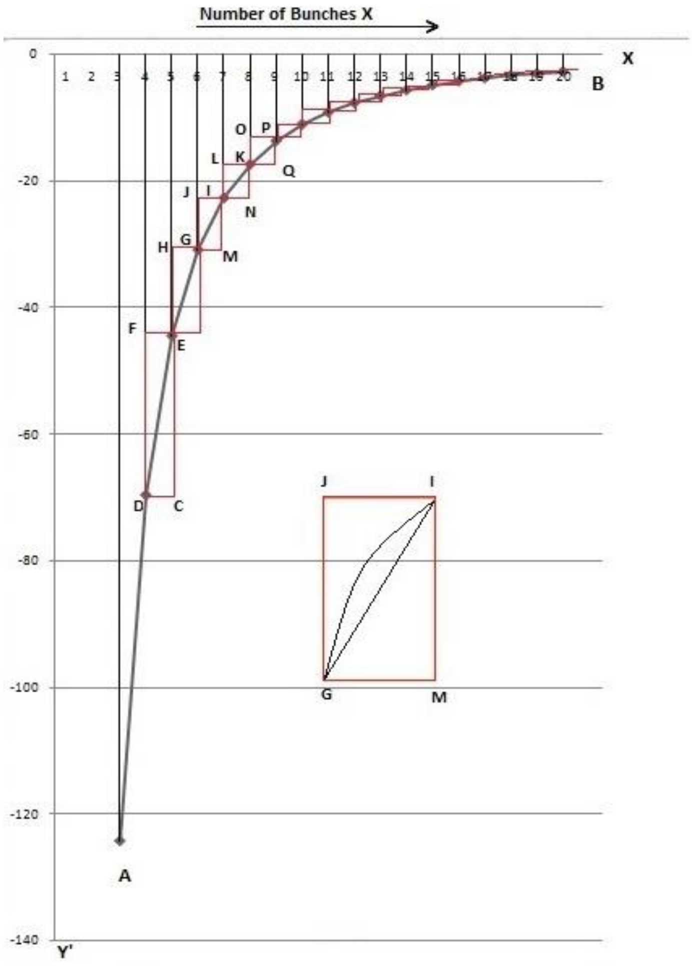

Fig.1 Plot of $\ln\left(1-\frac{1}{9x^2}\right)$ with x

Plot of $ln\,\{1-1/(9x^2)\}$ versus x, is a smooth curve as shown in black in Fig. 1, where number of bunches x are taken on X-axis and value of $ln\,\{1-1/(9x^2)\}$ on Y-axis. Quantities mentioned as $-20,-40,-60\ldots$ on negative Y-axis have multiplier 10^{-4} but it is not mentioned in the Fig. 1 for the sake of brevity. It is submitted that since x changes in steps from 1 to 2, 2 to 3, 3 to 4,…so on, therefore, actual graph must change in steps and that requires the value of $ln\,\{1-1/(9x^2)\}$ must also change in steps, resulting in a figure like stair steps as shown in red in Fig. 1. Since $\{1-1/(9x^2)\}$ is less than 1, value of $ln\,\{1-1/(9x^2)\}$ will always be negative, therefore, it is plotted on negative Y axis and x being positive is plotted on positive X-axis. If the areas of triangles above the curve are added algebraically, then smooth curve can approximate to stair steps figure. In that case , $\int \ln\left(1-\frac{1}{9x^2}\right)dx$ can be approximated to $\sum_{x=1}^{(n-1)/3} ln\{1-1/(9x^2)\}$. It is further submitted that portion of the curve shown as hypotenuse of the triangle between two consecutive bunches, is assumed as straight line for the purpose of calculation of area. In this way, area of individual triangle can be calculated as explained hereinafter.

Area of triangle DEF = half the area of rectangle $DCEF$. Or area of triangle DEF

$$=\frac{1}{2}\cdot ln\left\{\,1-\frac{1}{9\cdot(5^2)}\right\}-\frac{1}{2}\cdot ln\left\{1-\frac{1}{9\cdot(4^2)}\right\}.$$

Therefore, correction for *4^th^* to *5^th^* bunch

$$=\frac{1}{2}\cdot ln\left\{\,1-\frac{1}{9\cdot(5^2)}\right\}-\frac{1}{2}\cdot ln\left\{1-\frac{1}{9\cdot(4^2)}\right\}.$$

In this way, individual area of *2nd*, *3rd*, *4th* …so on up to *x*th bunch can be calculated.

Correction for 2nd bunch

$$=\frac{1}{2}\cdot ln\left\{\,1-\frac{1}{9\cdot(2^2)}\right\}-\frac{1}{2}\cdot ln\left\{1-\frac{1}{9\cdot(1^2)}\right\},$$

correction for 3rd bunch

$$=\frac{1}{2}\cdot ln\left\{\,1-\frac{1}{9\cdot(3^2)}\right\}-\frac{1}{2}\cdot ln\left\{1-\frac{1}{9\cdot(2^2)}\right\},$$

correction for 4th bunch

$$=\frac{1}{2}\cdot ln\left\{\,1-\frac{1}{9\cdot(4^2)}\right\}-\frac{1}{2}\cdot ln\left\{1-\frac{1}{9\cdot(3^2)}\right\},$$

——————- so on ——————————————————,

correction for *x*th bunch

$$=\frac{1}{2}\cdot ln\left\{\,1-\frac{1}{9\cdot(x^2)}\right\}-\frac{1}{2}\cdot ln\left\{\,1-\frac{1}{9\cdot(x-1)^2}\right\}.$$

Resultant correction for all bunches is algebraic sum of above mentioned each correction. On summing up, this equals $-(1/2)\cdot ln\,(8/9)+\,(1/2)\cdot ln\,\{1-1/(9x^2)\}$.

It is pertinent to state that correction for *1st* bunch is an initial condition and that equals $ln\ (8/9)$, therefore, it does not need correction and is not included while calculating resultant correction. Adding the resultant correction to correction already determined by equation (10), improved correction is given by relation.

$$ln\,C_x\simeq ln\frac{8}{9}+\frac{1}{9}\left(\frac{1}{x}-1\right)+\frac{1}{486}\left(\frac{1}{x^3}-1\right)+\frac{1}{10935}\left(\frac{1}{x^5}-1\right)-\frac{1}{2}\cdot ln\frac{8}{9}+\frac{1}{2}\cdot ln\,\left(1-\frac{1}{9x^2}\right).$$

On simplifying,

$$ln\,C_x\simeq\frac{1}{2}ln\frac{8}{9}+\frac{1}{2}ln\left(1-\frac{1}{9x^2}\right)+\frac{1}{9}\left(\frac{1}{x}-1\right)+\frac{1}{486}\left(\frac{1}{x^3}-1\right)+\frac{1}{10935}\left(\frac{1}{x^5}-1\right). \quad (11)$$

On taking antilog,

$$C_x\simeq\left\{\frac{8}{9}\left(1-\frac{1}{9x^2}\right)\right\}^{\frac{1}{2}}.\,exp\left\{\frac{1}{9}\left(\frac{1}{x}-1\right)+\frac{1}{486}\left(\frac{1}{x^3}-1\right)+\frac{1}{10935}\left(\frac{1}{x^5}-1\right)\right\}, \quad (12)$$

where $exp\left\{\frac{1}{9}\left(\frac{1}{x}-1\right)+\frac{1}{486}\left(\frac{1}{x^3}-1\right)+\frac{1}{10935}\left(\frac{1}{x^5}-1\right)\right\}$ is written in place of $e^{\left\{\frac{1}{9}\left(\frac{1}{x}-1\right)+\frac{1}{486}\left(\frac{1}{x^3}-1\right)+\frac{1}{10935}\left(\frac{1}{x^5}-1\right)\right\}}$.

C. Correction due to curvature of curve between successive bunches

For improving approximation, compensation has already been made by adding areas of half rectangles pertaining to each bunch but actual requirement is area above the curve, which we assumed as triangles. But in fact, these are not triangles as *the curve is not* *a straight line but has a curvature.* Our assumption made in para II *B* "Portion of the curve shown as hypotenuse of the triangle between two consecutive bunches is assumed as straight line for the purpose of calculation of area" yields error as the curvature has not been taken into consideration, while calculating areas. On inspection of the rectangle $GMIJ$ shown enlarged in Fig. 1, it is observed, area of triangle GJI was considered but in fact, the area enclosed by curve GI and straight lines GJ and IJ were to be considered. That means, there still exists an error due to curvature and that also needs correction. To compensate the error due to curvature, a quantity $\exp\{-.0178(1-1/x^3)\}$ is multiplied to equation (12) to further reduce the error. It is pertinent to explain that this assumed correction $\exp\{-.0178(1-1/x^3)\}$, when multiplied with correction given by equation (12), approximates best to actual correction as is evident from the data given in Table 1. That proves assumed additional correction approximates with the required correction due to curvature. On application of this correction due to curvature, equation (12) gets modified a

$$C_x \simeq \left\{\frac{8}{9}\left(1-\frac{1}{9x^2}\right)\right\}^{\frac{1}{2}}.exp\left\{\frac{1}{9}\left(\frac{1}{x}-1\right)+\left(\frac{1}{486}+.0178\right)\left(\frac{1}{x^3}-1\right)+\frac{1}{10935}\left(\frac{1}{x^5}-1\right)\right\} \quad (13)$$

D. Calculated correction multiplier using equation (13) versus ideal correction multiplier using equation (4)

Table 1 Ideal correction multiplier and calculated correction multiplier

Number of bunches	Ideal correction multiplier by equation (4)	Calculated correction multiplier by equation (11)
1	8/9	8/9
2	.8641975308642	.8641607472088
3	.85352842554489	.85351678483392
4	.84760114481194	.84759878677795
5	.84383402861277	.84383546137804
6	.84122960259853	.84123281665964
7	.83932205247926	.83932620106705
8	.83786489613815	.83786957800523
9	.83671556157555	.83672056824456
10	.83578587761824	.83589109259135
11	.83495672496188	.835023751722749
12	.83431246822965	.83437954378758
13	.8337639393222	.83383104299365
14	.8332912840278	.833335840285489
15	.83287978215914	.83294690788098
16	.83251828919814	.83258541649202
17	.83219821296238	.83226533838644
18	.83191282262872	.83197994396808
19	.83165677066731	.83172388651972
20	.83142575489768	.8314928644054
40000	.82699564030109	.8270013812
4000000	.82699336610472	.826999107

To check effectiveness of correction multiplier determined by equation (13), its values and those of ideal correction multipliers given by equation (4) for bunches 1 to 20 on lower side and 40000 and 4000000 on higher side, are given in the Table 1. The Table 1 shows, maximum error using equation (13) is less than .008 percent. Actual correction multipliers for bunches 40000 and 4000000 have been determined with the help of calculators as bunches being quite large, it is difficult to find their values using equation (4).

III. Factorial Tripling Formula

Referring to equation (5),

$$n! = \left(C_{\frac{n-1}{3}}\right)(3^{n-1})\left\{1\cdot 2\cdot 3\cdot \dots \cdot \left(\frac{n-7}{3}\right)\left(\frac{n-4}{3}\right)\left(\frac{n-1}{3}\right)\right\}^3$$

and writing number of bunches $(n-1)/3$ as x,

$$(3x+1)! \simeq C_x \cdot 3^{3x} \cdot (x!)^3, \quad (14)$$

where C_x is given by equation (13). On rearranging,

$$x! \simeq \frac{1}{3^x}.\left\{\frac{(3x+1)!}{C_x}\right\}^{1/3} \quad (15)$$

Table II Approximation of $(3x+1)!$ from given $x!$ and associated error

$x!$	$(3x+1)!$ according to formula (14)	$(3x+1)!$ actual	Percentage error
1!	24	$4! = 24$	0.00000
2!	5039.78548	$7! = 5040$	-0.00425
3!	3628750	$10! = 3628800$	$-1.36383401 \times 10^{-3}$
4!	6227003480	$13! = 6227020800$	$-2.78200897 \times 10^{-4}$
5!	$2.09228254 \times 10^{13}$	$16! = 20922789888000$	$1.69792307 \times 10^{-4}$
6!	$1.21645565 \times 10^{17}$	$19! = 1.216451 \times 10^{17}$	$3.82067047 \times 10^{-4}$
8!	$1.55112967 \times 10^{25}$	$25! = 1.551121 \times 10^{25}$	$5.58785444 \times 10^{-4}$
10!	$8.22387381 \times 10^{33}$	$31! = 8.22283865 \times 10^{33}$	$1.25887474 \times 10^{-4}$
12!	1.3763843×10^{43}	$37! = 1.37637531 \times 10^{43}$	$6.53260109 \times 10^{-4}$

15!	5.5026592×10^{57}	$46! = 5.50262216 \times 10^{57}$	6.7311102×10^{-4}
20!	$5.07583692 \times 10^{83}$	$61! = 5.07580214 \times 10^{83}$	$6.85254661 \times 10^{-4}$
500!	$7.22285883 \times 10^{4117}$	$1501! = 7.22808693 \times 10^{4117}$	$6.87128875 \times 10^{-4}$
5000!	$4.1202018 \times 10^{56133}$	$15000! = 4.12017321 \times 10^{56133}$	$6.94165027 \times 10^{-4}$

From Table II, it is clear that percentage error associated with formula (14) is few in thousand.

Example: Let us find out value of $7!$ when it is given, $2!$ equals 2. Here $x = 2$ and $3x + 1$ is 7. Then $7! \simeq (C_2)\{3^{3(2)}\}(2!)^3$ using equation (14). Correction multiplier for two bunches is applied since number of terms 7 makes 2 bunches excluding first term 1, $C_2 = (.8641607472088)$, using equation (13). Therefore, $7! \simeq C_2.\{(2!).3^2\}^3 \simeq (.8641607472088).(5832) \simeq 5039.7854777217$. Actual value of $7! = 5040$. Error is $-.004256$ percent.

That proves if $x!$, where integer $x > 0$, is given, then $(3x + 1)!$ can be approximated to $C_x \cdot 3^{3x} \cdot (x!)^3$, where C_x is given by equation (13).

A. Factorial tripling formula when $x \to \infty$

When $x \to \infty$ or is extremely large, $1/x$ can be neglected, then equation (13) transforms to

$$C_\infty \simeq \frac{2}{3} \cdot \sqrt{2} \cdot exp\left\{-\frac{1}{9} - \left(\frac{1}{486} + .0178\right) - \frac{1}{10935}\right\},$$

where C_∞ is correction multiplier for very very large bunches and has constant value of $.8269990839956686$. On substituting this value of C_∞ in equation (14), we get $(3x + 1)! \simeq C_\infty \cdot \{3^x \cdot (x!)\}^3$ or

$$(3x + 1)! \simeq (.8269990839956686) \cdot \{3^x \cdot (x!)\}^3 \quad (16)$$

IV. Approximation Of Factorial

Examination of factorial tripling formula given by relation

$$x! \simeq \frac{1}{3^x} \cdot \left\{\frac{(3x + 1)!}{C_x}\right\}^{1/3},$$

reveals that x appears in right hand side as well as left hand side, therefore, value of $x!$ can help approximate the value of $(3x + 1)!$ and on approximation of value $(3x + 1)!$, value of $(9x + 4)!$ can be approximated.

$$x! \simeq \frac{1}{3^x} \cdot \left\{\frac{(3x+1)!}{C_x}\right\}^{\frac{1}{3}} \simeq \left(\frac{1}{3^x}\right)\left[\left(\frac{1}{C_x}\right)\left(\frac{1}{3^{3x+1}}\right)\left\{\frac{(9x+4)!}{C_{3x+1}}\right\}^{\frac{1}{3}}\right]^{\frac{1}{3}},$$

where C_{3x+1} is a correction multiplier for $3x + 1$ bunches. From $(9x + 4)!$, value of $(27x + 13)!$ can be approximated and so on. If given x is 1, value of $4!, 13!, 40! \dots n!$ can be approximated, when n is given by telescopic series

$$n = 1 + 3^1 + 3^2 + \cdots + 3^k$$

and k is an integer $1, 2, 3, \dots$

A. *Geometrical progression with first term* 1 *and common ratio* 3

Consider a geometrical progression GP or telescopic series with common ratio 3 and first term 1. That is $n = 1 + 3^1 + 3^2 \dots 3^k$, where n is its sum, $k + 1$ are number of terms of this GP and k can have any value $1, 2, 3, \dots$ Value of k equal to 0, is not included as it leads to n equal to 1, which has value of its factorial as 1, requiring no calculation.

On summing up the series, n can be written

$$n = \frac{1}{2} \cdot (3^{k+1} - 1) = 1 + 3^1 + 3^2 \dots 3^k \quad (17)$$

Or

$$2n + 1 = 3^{k+1} \quad (18)$$

If value of factorial x is given and x is a positive integer other than 1, then factorial n, when n has any value of the forms $(3x + 1)$, $(9x + 4)$, $(27x + 13)$, ... so on upto $\{3^k x + (3^k - 1)/2\}$, can be approximated. Values of n, when given $x = 1$ and k varies from 4 to 15, are mentioned in Table III as illustrations.

Table III Number of terms k of GP and its sum (n)

k	n	k	n	k	n
1	4	6	1093	11	265720
2	13	7	3280	12	797161
3	40	8	9841	13	2391484
4	121	9	29524	14	7174453
5	364	10	88573	15	21523360

These are the values of n for which their factorials can be approximated, if given factorial is 1.

B. Recursive nature of Factorial Tripling Formula

If n is given by equation (17), formula for $n!$ can be derived using recursive nature of factorial tripling formula of equation (14). Applying this formula successively when given $x = 1$,

$$4! \simeq 3^3.C_1$$

$$13! \simeq C_4.3^{3(4)}.(3^3.C_1)^3 \simeq C_4.C_1^3.3^{3(4)}.3^{3^2}$$

$$40! \simeq C_{13}.3^{3(13)}.\{C_4.3^{3(4)}.(3^3.C_1^3)^3\}^3$$

$$\simeq C_{13}^{;}.C_4^3.C_1^{3^2}.3^{40-1}.3^{40-4}.3^{3^3}$$

$$121! \simeq \left[C_{40}^3.3^{120}.\{C_{13}^3.C_4^{3^2}.C_1^{3^3}.3^{3(13)}.3^{3^2(4)}.3^{3^3(1)}\}^3\right]$$

$$\simeq C_{40}.C_{13}^3.C_4^{3^2}.C_1^{3^3}.3^{121-1}.3^{121-4}.3^{121-13}.3^{81}$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots so\ on.$$

By mathematical induction,

$$n! \simeq \left(C_{\frac{n-1}{3}} \cdot C_{\frac{n-4}{3^2}}^3 \cdot C_{\frac{n-13}{3^3}}^{3^2} .. C_1^{3^{k-1}}\right) \cdot 3^{\{(n-1)+(n-4)+(n-13)+\dots+3^k\}}$$

$$\simeq \left(C_{\frac{n-1}{3}} \cdot C_{\frac{n-4}{3^2}}^3 \cdot C_{\frac{n-13}{3^3}}^{3^2} \dots C_1^{3^{k-1}}\right) \cdot 3^{\left\{k\left(n+\frac{1}{2}\right)-\frac{3}{4}(3^k-1)\right\}}$$

Let cumulative correction multiplier be C given by relation

$$C = \left(C_{\frac{n-1}{3}} \cdot C_{\frac{n-4}{3^2}}^3 \cdot C_{\frac{n-13}{3^3}}^{3^2} \dots C_1^{3^{k-1}}\right),$$

and on substituting 3^k with $(2n + 1)/3$ using equation (18),

$$n! \simeq \left(\frac{2n+1}{3}\right)^{n+1/2} \cdot 3^{\left(\frac{1-n}{2}\right)}$$

On simplification ,

$$n! \simeq C \cdot \sqrt{\frac{(2n+1)^{2n+1}}{3^{3n}}} \tag{19}$$

Hence if n is given by equation (18) i.e. $n = (3^{k+1} - 1)/2$ where k is any positive integer $1, 2, 3, \ldots$, then $n!$ can be approximated using equation (19) provided n must be an integer $4, 13, 40, 121, \ldots, (3^{k+1} - 1)/2$.

Notwithstanding approximation of factorial of these integers, if value of $x!$ is given, approximation of factorial of integers $(3x+1)!$, $(9x+4)!, (27x+13)!, \ldots \{3^k x + (3^k - 1)/2\}$ can also be found, where k is any integer 1, 2, 3,… That means *factorial of any integer of the form* $\{3^k x + (3^k - 1)/2\}$ *can be approximated if* $x!$ *is given and* k *is any integer* $1, 2, 3, \ldots$

C. Error associated with factorial approximation formula (15) as compared to Stirling Formula

Percentage errors associated with factorials when computed using equation (14) and percentage errors associated, when computed, using Stirling formula $n! \simeq \sqrt{2n \cdot \pi} \cdot (n/e)^n$, are given in Table IV.

Table IV Comparison of errors equation (15) and Stirling Formula

n	Approximation of $n!$ using formula (14)	Percentage error using formula (14)	Approximation of $n!$ using Stirling formula	Percentage error using Stirling formula
4	24	0.000	23.50617513	−02.57
13	622700348	$-2.78200897 \times 10^{-4}$	6.18723948×10^{9}	−0.638850039
40	$8.15913874 \times 10^{47}$	$-1.7268532 \times 10^{-4}$	$8.14217264 \times 10^{47}$	−0.2081121396
121	$8.09431269 \times 10^{200}$	$-1.74998877 \times 10^{-4}$	$8.08872587 \times 10^{200}$	−0.0688466564

It is clear from Table IV that formula (14) yields percentage error few in ten thousands whereas Stirling formula yields few in hundred. Formula (19) has better accuracy than that of Stirling Formula.

V. Results And Conclusions

Overview of the paper makes amply clear that multiplying terms, $2, 3, \ldots, (n-2), (n-1), (n)$ of $n!$ excluding 1 , are grouped in three consecutive integers, forming $(n-1)/3$ bunches. In such arrangement, a bunch has consecutive integers $a-1$, a, and $a+1$, where a is of the form $3b$ and integer $b \geq 1$. Geometric mean of the terms in the bunch is $(a^3 - a)^{1/3}$ and their arithmetic mean is $(1/3)\{(a-1) + (a) + (a+1)\}$ or a. It is obvious, AM 'a' of the bunch is more than GM $(a^3 - a)^{1/3}$. Our endeavour is to approximate AM to GM. It is accomplished by discovering a function C_x called correction factor or multiplier and this function C_x varies with x, where integer $x \geq 1$ is number of bunches containing three consecutive integers. Each bunch can, then be replaced with cube of its AM. Factorial of integer n has $(n-1)/3$ bunches and each bunch is replaceable with cube of its AM multiplied with its correction factor. Product of each $(AM)^3$ and their correction factors, yields approximation to value of factorial. It is also observed that each multiplying $(AM)^3$ have a common multiplier 3^3. Since there are $(n-1)/3$ such AM's, therefore, that gives rise to cumulative multiplier of 3^{n-1}. Mathematically , $n!$, then can be given by relation

$$n! \simeq \left(C_{\frac{n-1}{3}}\right) \cdot (3^{n-1}). \left\{\left(\frac{n-1}{3}\right)!\right\}^3$$

Assuming $(n-1)/3 = x$, then above said relation can be written $(3x+1)! \simeq C_x . 3^{3x} . (x!)^3$, where C_x is a correction factor for x bunches. The value of C_x is given by relation

$$C_x = \left[\left(1 - \frac{1}{3^2}\right) \cdot \left(1 - \frac{1}{6^2}\right) \cdot \left(1 - \frac{1}{9^2}\right) \ldots \left\{1 - \frac{1}{(3x)^2}\right\}\right]$$

and this relation on taking logarithm of both sides can be written

$$\ln C_x = \sum_{x=1}^{(n-1)/3} \ln\{1 - 1/(9x^2)\}.$$

which can be approximated to $\int \ln\left(1 - \frac{1}{9x^2}\right) dx$. Assuming x to be large, expanding $ln\left(1 - \frac{1}{9x^2}\right)$, integrating it and, then taking its antilog, yields

$$C_x = \frac{8}{9} . exp\left\{\frac{1}{9}\left(\frac{1}{x} - 1\right) + \frac{1}{486}\left(\frac{1}{x^3} - 1\right) + \frac{1}{10935}\left(\frac{1}{x^5} - 1\right)\right\}.$$

Value of C_x still needs further corrections on two counts. First x varies in steps of 1 to 2, 2 to 3, 3 to 4, so on whereas C_x given by above said equation when plotted, provides smooth curve. Second, the plot of $ln\left(1 - \frac{1}{9x^2}\right)$ with x is not a straight line but has a curvature between between steps. On application of corrections due to above said reasons, resultant correction obtained is

$$C_x \simeq \left\{\frac{8}{9}\left(1 - \frac{1}{9x^2}\right)\right\}^{\frac{1}{2}} . exp\left\{\frac{1}{9}\left(\frac{1}{x} - 1\right) + \left(\frac{1}{486} + .0178\right)\left(\frac{1}{x^3} - 1\right) + \frac{1}{10935}\left(\frac{1}{x^5} - 1\right)\right\}$$

Application of this resultant correction, provides improved approximation to Factorial Tripling Formula

$$(3x+1)! \simeq C_x . 3^{3x} . (x!)^3.$$

This formula is recursive in nature and if applied successively, can approximate factorial of a positive integer n, if n is given by relation

$$n = 3^0 + 3^1 + 3^2 + 3^3 \ldots + 3^k = (3^{k+1} - 1)/2,$$

where k is a positive integer $1, 2, 3, \ldots$ In that case,

$$n! \simeq C \cdot \sqrt{\frac{(2n+1)^{2n+1}}{3^{3n}}}$$

where C is cumulative correction multiplier given by relation

$$C = \left(C_{\frac{n-1}{3}} \cdot C^3_{\frac{n-4}{3^2}} \cdot C^{3^2}_{\frac{n-13}{3^3}} \ldots C_1^{3^{k-1}}\right)$$

and $C_{\frac{n-1}{3}}, C_{\frac{n-4}{3^2}}, C_{\frac{n-13}{3^3}}, \ldots C_1$ are correction multipliers for $(n-1)/3$, $(n-4)/3^2$, $(n-13)/3^3, \ldots, 1$ bunches. In addition to above, if value of $x!$, where integer $x \geq 1$, is given, approximation of factorial of any integer of the form $\{3^k x +$

$(3^k - 1)/2\}$, where integer $k > 0$, can be made, using recursive relation $(3x + 1)! \simeq C_x . 3^{3x} . (x!)^3$.

ACKNOWLEDGEMENTS

We acknowledge the help provided by the scientific calculators at websites https://www.desmos.com and https://keisan.casio.com/calculator in calculating the values of tedious and large exponential, logarithmic and factorial terms.

REFERENCES

Gert Almkvist and Bruce Berndt (1988). Gauss, Landen, Ramanujan, the Arithmetic- Geometric Mean, Ellipses, π, and the Ladies Diary. *The American Mathematical Monthly*

Carl Barratt and Ramesh Sharma (2012). An inductive proof of the condition for the AM-GM equality. *The Mathematical Gazette* Vol. 96, No. 535, pp. 131-133

J. M. Aldaz (2011). Comparison of differences between arithmetic and geometric means, *Tamkang Journal of Mathematics*, Volume 42, Number 4, 453-462.

Li, Y.; Gu, X.-M.; Zhao, J. (2018). The Weighted Arithmetic Mean–Geometric Mean Inequality is Equivalent to the Hölder Inequality. *Symmetry*, 10, 380.

Prithwijit De (2016). The Arithmetic Mean – Geometric. Mean – Harmonic Mean: Inequalities and a Spectrum of Applications, *Resonance – Journal of Science Education*, Volume 21, Issue 12, pp 1119-1133

Weisstein, Eric W. "Stirling's Approximation. From *MathWorld --A Wolfram Web Resource* https://mathworld.wolfram.com/StirlingsApproximation.html

William H. Jean and Billy P. Helms (1983). Geometric Mean Approximations, *The Journal of Financial and Quantitative Analysis.*

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