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On some properties and application of the transformed version of a probability model defined on the unit interval

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***Abstract*—Through this paper, the exponentiation transformation has been applied to a version of the Unit Lindley distribution having support on $(0, 1)$. The graphical behaviour of the density curve for different values of the parameters is studied and various statistical properties of the distribution such as descriptive statistics, generating functions, reliability properties, distributions of the order statistics are discussed. Random numbers from the proposed distribution are generated and a simulation study is performed to assess the behaviour of the parameter estimates on the basis of the generated sample. The parameters of the distribution are estimated using the maximum likelihood method. Finally, the utility of the developed model is exhibited through an original data set on timing of infant deaths.**

***Index Terms*—Exponentiation, Maximum Likelihood estimation, Statistical Properties, Simulation Study, Unit Lindley distribution.**

I. Introduction

With the advent of time, the volume of data available for analysis is increasing at quite a fast pace. Distribution theory plays a vital role in modelling data arising from several areas such as queuing theory, reliability theory, life insurance, demography etc. (Poonia and Azad (2021) , Singh et al. (2021)). Some of the existing conventional distributions may fail to provide a satisfactory fit in case of some real life data sets. For instance, data coming a population having a non-monotone hazard rate can not be modeled using some conventional distributions which have increasing or decreasing hazard rate functions. Therefore, the existing distributions are usually extended by adding new parameters to it.

A number of lifetime models are available in the literature for analyzing survival or lifetime data. But owing to the specialized nature of some data sets, the conventional lifetime models may fail to appropriately model it. The usual procedure is to add an extra parameter by means of generators or present distributions are compounded or mixed to obtain more flexible models (Salahuddin et al., 2021).

The exponentiated family of distributions was introduced by Gupta et al. (1998). The cumulative distribution function (cdf) of the exponentiated distribution is constructed by raising the cdf of a baseline distribution (say $F(x)$) to an arbitrary power, say $\alpha > 0$, so that the cdf of the exponentiated distribution becomes $G(x) = [F(x)]^{\alpha}$. The moments of the exponentiated distribution is strictly larger or smaller than those of the baseline distribution (Cordeiro et al., 2013). The exponentiated versions of several commonly used distributions have been derived so far, some of the popular ones being Exponentiated Exponential distribution (Gupta & Kundu, 2001), Exponentiated Weibull distribution (Mudholkar & Srivastava, 1993), Exponentiated Fréchet distribution (Nadarajah & Kotz, 2006) and Exponentiated Gumbel distribution (Nadarajah, 2006). One may refer to Ali et al. (2007) for getting an extensive insight into the exponentiated distributions. The new Unit Lindley distribution is an alternative distribution to Beta and Uniform distribution to model data occurring in the range $(0, 1)$ (Mazucheli et al., 2020). It has been obtained by transforming a Lindley distributed random variable, which is indexed by only one parameter. This distribution has several advantages over the Beta distribution. Another version of the Unit Lindley distribution also exists in literature, published by Mazucheli et al. (2019), which is the derivative of the popular Lindley distribution. The exponentiated version of this Unit Lindley distribution has also been established (Irshad et al., 2021). However, no researcher has so far attempted to derive the exponentiated version of this new unit Lindley distribution arising in the unit interval (Mazucheli et al., 2020). Hence, this paper attempts to develop the exponentiated version of the new Unit Lindley distribution. The remaining paper is organized as follows: Section 2 contains the derivation of the pdf and cdf of the proposed distribution. The statistical properties of the distribution are explored in Section 3. Section 4 shows the derivation of the pdf of the smallest and largest order statistic from the proposed distribution. Section 5 deals with the

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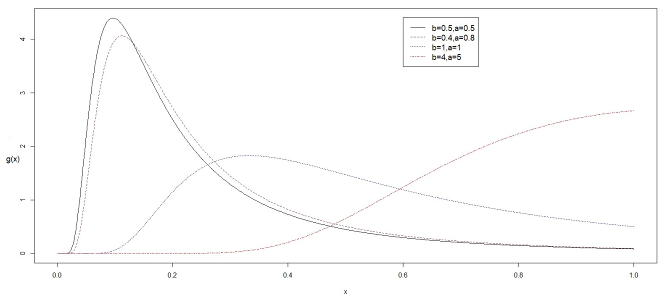

Figure 1. pdf plot of the Exponentiated new Unit Lindley distribution for different values of θ and a

simulation study and the maximum likelihood estimation of the parameters of the distribution on the basis of the generated sample. Section 6 shows the utility of the distribution with the help of a real life data set. Section 7 finally summarizes the findings of the paper.

II. Exponentiated version of the latest Unit Lindley distribution: derivation of pdf and cdf

The umulative distribution function of the latest version of the Unit Lindley distributed r.v. X is given by (Mazucheli et al., 2020)

$$F(x \mid \theta) = \frac{\theta + x}{x(1+\theta)} e^{-\theta\left(\frac{1-x}{x}\right)}, 0 < x \leq 1; \theta > 0 \qquad (1)$$

The distribution function of exponentiated class of distributions defined by Mudholkar & Srivastava (1993) is as shown below

$$F(x) = G^a(x); a > 0$$

Therefore, the cdf of the exponentiated version of the latest form of Unit Lindley distribution is given by

$$\begin{aligned} G(x) &= \{F(x \mid \theta)\}^a ; a > 0 \\ &= \left\{\frac{\theta + x}{x(1+\theta)}\right\}^a e^{-a\theta\left(\frac{1-x}{x}\right)} \qquad (2) \end{aligned}$$

The probability density function (pdf) corresponding to the cdf in (2) is finally obtained as

$$\begin{aligned} g(x) &= \frac{d}{dx} G(X) \\ &= \frac{a\theta^2}{x^{a+2}} \frac{(\theta + x)^{a-1}}{(1+\theta)^a} e^{-a\theta\left(\frac{1-x}{x}\right)}; a, \theta > 0; 0 < x \leq 1 \end{aligned} \qquad (3)$$

The pdf plot of the Exponentiated version of the latest Unit Lindley distribution for different values of θ and a is exhibited in figure 1. It is evident from figure 1 that the skewness and kurtosis of the proposed distribution decreases with an increase in the value of both a and θ.

III. Statistical properties

This section presents some important properties of the Exponentiated version of the latest Unit Lindley distribution.

A. Moments and other descriptive statistics

The r^{th} moment about the origin of a random variable X, denoted by $\mu_r^{'}$ is the expected value of X^r.
Symbolically,

$$\begin{aligned} \mu_r^{'} &= E(X^r) \\ &= \int_{-\infty}^{\infty} x^r f(x) dx \end{aligned}$$

Therefore, the r^{th} moment of the Exponentiated version of the latest form of the Unit Lindley distribution is,

$$\begin{aligned} \mu_r^{'} &= \int_0^1 x^r \frac{a\theta^2}{x^{a+2}} . \frac{(\theta + x)^{a-1}}{(1+\theta)^a} . e^{-a\theta\left(\frac{1-x}{x}\right)} dx \\ &= e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k} E_i(r+k-a, a\theta) \end{aligned}$$

where $E_i(r+k-a, a\theta) = \int_1^{\infty} u^{-(r+k-a)} e^{-a\theta u} du$ is referred to as the Exponential integral function (Abramowitz and Stegun, 1974)
The first four raw moments as shown below are obtained by putting the value of r=1,2,3,4 in the expression for $\mu_r^{'}$.

$$\begin{aligned} \mu_1^{'} &= e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k} E_i(1+k-a, a\theta) \\ \mu_2^{'} &= e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k} E_i(2+k-a, a\theta) \\ \mu_3^{'} &= e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k} E_i(3+k-a, a\theta) \\ \mu_4^{'} &= e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k} E_i(4+k-a, a\theta) \end{aligned}$$

The skewness (γ_1) and kurtosis (β_2) coefficient of the proposed distribution can be obtained by using the following formulae:

$$\gamma_1 = \frac{\mu_3}{\mu_2{}^{\left(\frac{3}{2}\right)}}$$

and

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

where the central moments are obtained from the raw moments with the help of the following relationships:

$$\begin{aligned} \mu_1 &= \mu_1^{'} - \mu_1^{'} = 0 \\ \mu_2 &= \mu_2^{'} - \mu_1^{'2} \\ \mu_3 &= \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3} \\ \mu_4 &= \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3\mu_1^{'4} \end{aligned}$$

In particular,

$$E(X) = e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k} E_i(1+k-a, a\theta)$$

and

$$V(X) = e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \left[\sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k} \right.$$
$$E_i(2+k-a, a\theta) - e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \left\{ \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k} \right.$$
$$\left. E_i(2+k-a, a\theta) \right\}^2$$

The plots of skewness and kurtosis measures for different values of a (keeping θ fixed) and different values of θ (keeping a fixed) are presented in Figure 2 and 3 respectively.
It is apparent from figure 2 that the skewness decreases with increase in the values of both a and θ while keeping the other parameter in either case as fixed. From figure 3, it is clear that kurtosis decreases with an increase in the value of θ when a is kept fixed, whereas the kurtosis surges with an increase in the value of a when θ is kept fixed.

B. Generating functions

The moment generating function of the Exponentiated version of the latest Unit Lindley distribution is given by

$$M_X(t) = E\left(e^{tX}\right), t \in \mathbb{R}$$
$$= e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k}$$
$$\int_0^1 x^{k-a-2} e^{tx - \frac{a\theta}{x}} dx \tag{4}$$

The moments of the distribution presented under Section (III.A) can also be obtained from $M_X(t)$ by virtue of the following relation:

$$E(X^r) = \frac{d^r}{dt^r} M_X(t) |_{t=0}$$

provided the r^{th} moment exists.
The characteristic function of the Exponentiated new Unit Lindley distribution can be obtained as shown below:

$$\phi_X(t) = E\left(e^{itX}\right), t \in \mathbb{R}, i = \sqrt{-1}$$
$$= e^{a\theta} \frac{a\theta^2}{(1+\theta)^a} \sum_{k=0}^{\infty} \binom{a-1}{k} \theta^{a-1-k}$$
$$\cdot \int_0^1 x^{k-a-2} e^{itx - \frac{a\theta}{x}} dx \tag{5}$$

The moments of the distribution presented under Section (III.A) can be obtained from $\phi_X(t)$ also by virtue of the following relation:

$$E(X^r) = \frac{d^r}{dt^r} \phi_X(t) |_{t=0}$$

provided the r^{th} moment exists.

C. Reliability properties

The hazard rate function is a very important characteristic of a probability distribution, which measures the variation in failure rate over the lifetime of a system (Maurya, 2021). The hazard rate function of the proposed distribution is obtained as

$$\begin{aligned} h(t) &= \frac{g(t)}{1-G(t)} \\ &= \frac{\frac{a\theta^2}{t^{a+2}} \frac{(\theta+t)^{a-1}}{(1+\theta)^a} e^{-a\theta\left(\frac{1-t}{t}\right)}}{1 - \left\{\frac{\theta+t}{t(1+\theta)}\right\}^a e^{-a\theta\left(\frac{1-t}{t}\right)}} \end{aligned} \tag{6}$$

Figure 4 displays the hazard rate function plot of the Exponentiated new Unit Lindley distribution. It is clear from the figure that when θ is kept fixed and value of a is smaller then 1, then the hazard rate function is an increasing function of time, but for values of $a \geq 1$, it increases initially after which it remains constant over time. When value of a is kept fixed at values $>$ or $<$ 1, the hazard rate behaves as an increasing function of time, irrespective of what the value of θ is.
We shall now discuss some order relationships of the Exponentiated new Unit Lindley distribution: A r.v. Y is said to be smaller than another r.v. X

1) Stochastically, i.e., $(Y \leq_{ST} X)$ if $F_Y(t) \geq F_X(t)$
2) in Likelihood Ratio, i.e., $(Y \leq_{LR} X)$ if $\frac{f_Y(t)}{f_X(t)}$ is a decreasing function of t.

Consider two random variables X and Y such that $X \sim$ Exponentiated new Unit Lindley distribution with parameters a and θ and $Y \sim$ new Unit Lindley distribution with parameter θ, each of which is having support in $(0,1)$. Now, $G_X(x) = \{F_Y(x)\}^a, a > 0$ where $G_X(.)$ and $F_Y(.)$ represent the distribution functions of X and Y respectively. Thus,

1) X is stochastically smaller than Y if $a < 1$
2) Y is stochastically smaller than X if $a > 1$

Again, if $g_X(.)$ and $f_Y(.)$ represent the density functions of X and Y respectively, then we have the following relationship between them: $g_X(x) = a\{F_Y(x)\}^{a-1} f_Y(x)$ Hence,

1) X is smaller in likelihood ratio than Y if $\frac{1}{\alpha\{F(x)\}^{a-1}}$ is a decreasing function of x, i.e., when $a > 1$.
2) Y is smaller in likelihood ratio than X if $\alpha\{F(x)\}^{a-1}$ is a decreasing function of x, i.e., when $a < 1$.

Thus, we see that the order property between the baseline distribution and its exponentiated version depends only on the value of the parameter a.

IV. Order statistics

Consider a random sample $y_1, y_2, \ldots, y_n$ drawn from the Exponentiated new Unit Lindley distribution. Further, let $y_{1:n} < y_{2:n} < \ldots, y_{n:n}$ be the order statistics corresponding to this sample. Denoting the population p.d.f. and c.d.f. by $g(y)$ and $G(y)$ respectively, the p.d.f. $g_p(y)$ of the p^{th} order statistic $y_{p:n}$ is obtained as (Maurya, 2021)

$$g_p(y) = \frac{n!}{(p-1)!(n-p)!} \sum_{i=0}^{n-p} (-1)^i \binom{n-p}{i} G^{p-i+1}(y) g(y)$$

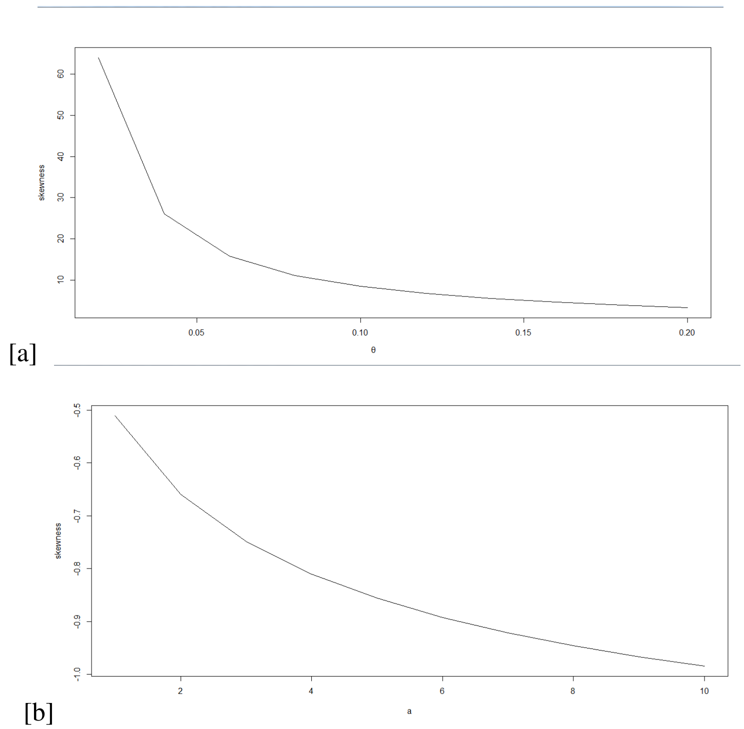

Figure 2. Skewness plot for (a) different values of θ (keeping a fixed at 10) and (b) different values of a (keeping θ fixed at 0.8)

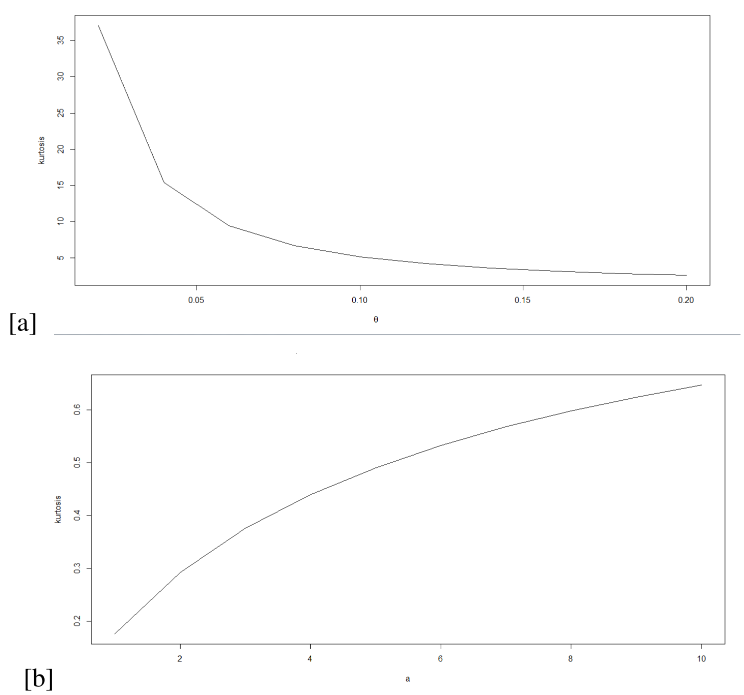

Figure 3. Kurtosis plot for (a) different values of θ (keeping a fixed at 10) and (b) different values of a (keeping θ fixed at 0.8)

$$= \frac{n!}{(p-1)!\,(n-p)!} a\theta^2 \left[\sum_{i=0}^{n-p} (-1)^i \begin{pmatrix} n-p \\ i \end{pmatrix} \right.$$

$$\left. \frac{(\theta+y)^{ap+ai-1}}{y^{ap+ai+2}\,(1+\theta)^{ap+ai}} \exp\left\{ -\left(ap+ai-a\right)\theta \left(\frac{1-y}{y}\right) \right\} \right]$$

and the corresponding c.d.f. is

$$G_p\left(y\right) = \sum_{l=p}^{n} \sum_{j=0}^{n-i} \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} n-l \\ j \end{pmatrix} (-1)^j \, G^{l+j}\left(y\right)$$

$$= \sum_{l=p}^{n} \sum_{j=0}^{n-i} \left(\begin{array}{c} n \\ l \end{array} \right) \left(\begin{array}{c} n-l \\ j \end{array} \right) (-1)^j \frac{(\theta+y)^{a(l+j)}}{\{y(1+\theta)\}^{a(l+j)}} \exp\left\{ -a\theta(l+j)\left(\frac{1-y}{y}\right)\right\}$$

In particular, the p.d.f. of the smallest order statistic $y_{1:n}$ is

$$g_1(y) = na\theta^2 \sum_{i=0}^{n-1} (-1)^i \left(\begin{array}{c} n-1 \\ i \end{array} \right) \frac{(\theta+y)^{ai+a-1}}{y^{a+ai+2}(1+\theta)^{a+ai}} \exp\left\{ -(ai\theta)\left(\frac{1-y}{y}\right)\right\} \Bigg]$$

and that of the largest order statistic $y_{n:n}$ is

$$g_n(y) = na\theta^2 \frac{(\theta+y)^{na-1}}{y^{na+2}(1+\theta)^{na}} \exp\left\{ -(n-1)a\theta\left(\frac{1-y}{y}\right)\right\}$$

Figure 5 shows the density plots of the smallest and largest order statistic from the Exponentiated version of the latest form of the Unit Lindley distribution for different set of values of a and θ. It is evident from figure 5 that the $g_1(y)$ of the smallest order statistic $y_{1:n}$ is a decreasing function of $y_{1:n}$ and $g_n(y)$ of the largest order statistic $y_{n:n}$ is a decreasing function of $y_{n:n}$. Further, the peakedness of both $g_1(y)$ and $g_n(y)$ increases with an increase in θ.

V. Simulation and Estimation

A. Simulation

This section presents a Monte Carlo simulation study conducted for the purpose of evaluation of the finite-sample behavior of the maximum likelihood estimates of the Exponentiated version of the latest Unit Lindley distribution. Samples of size $n = 50, 100, 250, 500$ and 1000 are generated and 1000 replications is considered for each sample size. For simulating n observations from the proposed distribution, the following algorithm was implemented:

Step 1. n random numbers from $U(0,1)$ were generated, say $U_i; i = 1, 2, \dots n$

Step 2. For each 'i', the equation $G(y_i) = U_i$ was solved.

Step 3. The value of y_i obtained by solving $y_i = G^{-1}(U_i)$ is a random value from the Exponentiated version of the latest form of the Unit Lindley distribution.

Here, $G^{-1}(.)$ is the quantile function of the proposed distribution. The performance evaluation of the maximum likelihood estimates was done based on the estimated Bias and Mean Square Error (MSE) measures. Table I presents the simulation results for the Exponentiated new Unit Lindley distribution.

Table I

Simulation results for Exponentiated new Unit Lindley distribution for $a = 0.5$ and $\theta = 1.2$

Sample Size (n)	Parameter	ML estimate	Bias	MSE
50	a	0.12	0.433	0.188
	θ	0.75	0.699	0.489
100	a	0.19	0.318	0.101
	θ	0.82	0.379	0.143
250	a	0.36	0.281	0.079
	θ	0.89	0.312	0.097
500	a	0.42	0.054	0.002
	θ	1.04	0.159	0.025
1000	a	0.513	0.013	0.0001
	θ	1.214	0.014	0.0002

B. Estimation: Maximum likelihood method

The maximum likelihood estimates are obtained by maximizing the log-likelihood function, which is given by:

$$\log L = n\log a + 2n\log\theta - (a+2)\log x + (a-1)\log(\theta+x) - a\log(1+\theta) - a\theta\left(\frac{1-x}{x}\right)$$

Since the maximum likelihood equations are non-linear in nature, numerical techniques are employed to obtain solutions to these equations.

We observe from Table I that the magnitude of bias and the MSE values of both a and θ decrease and approach to zero as the sample size increases. This indicates towards the consistency of the maximum likelihood estimates of the parameters of the proposed distribution.

VI. Application

In this section, a real-life data set has been studied for showing the application of the proposed distribution. Furthermore, the baseline new Unit Lindley model is also applied to the data set for model comparison. The data set has been reported in Singh et al. (2015).

A. Data Set

The data set comprises of the number of infant deaths distributed according to months in Uttar Pradesh during the year 2005-06. Here, we consider one month as $1/12^{th}$ of a year, two months as $2/12^{th}$ of a year and so on, so that the unit of measurement is in years and the data points assume values in $(0,1)$. By infant deaths, we mean death of children who die in the age interval $(0,1)$ years, i.e., before completing the first year of life. The maximum likelihood estimates of the parameters, AIC value, Kolmogorov-Smirnov one sample statistic and the p-value of the Kolmogorov-Smirnov one sample test corresponding to the Exponentiated version of the latest Unit Lindley and Unit Lindley distribution fitted to the data set under consideration is displayed in table II below:

Table II clearly shows that for the considered data set on timing of infant deaths, the Exponentiated new unit Lindley

Table II
MAXIMUM LIKELIHOOD ESTIMATES OF THE PARAMETERS, LOG-LIKELIHOOD (LL) AND AIC VALUE, KOLMOGOROV-SMIRNOV ONE SAMPLE STATISTIC AND THE p-VALUE OF THE KOLMOGOROV-SMIRNOV ONE SAMPLE TEST FOR THE DATA SET UNDER CONSIDERATION

Distribution	Parameter estimates	LL	AIC	K-S Statistic	p-value
EUL	$\hat{a} = 0.4, \hat{\theta} = 10$	-20.436	44.872	0.217	0.9135
UL	$\hat{\theta} = 0.222$	-23.985	49.97	0.406	0.7416

distribution has larger log-likelihood values and smaller AIC value and Kolmogorov-Smirnov statistic value as compared to the baseline new Unit Lindley distribution. So, we can conclude that the Exponentiated version of the latest Unit Lindley distribution is a better fit to the data set under consideration and has better modelling potential than the new Unit Lindley distribution.

VII. CONCLUSION

A new distribution viz. the Exponentiated new Unit Lindley distribution has been proposed in this paper. Its pdf and cdf have been derived and various statistical properties such as moments, skewness, kurtosis, generating functions and reliability properties have been explored. The hazard rate function, when θ is kept fixed and value of a is smaller then 1, behaves as an increasing function of time, but for values of $a \geq 1$, it increases initially after which it remains constant over time. Whereas when value of a is kept fixed at values $>$ or $<$ 1, the hazard rate behaves as an increasing function of time, not being affected by the value of θ. The order relationship between the Unit Lindley and its exponentiated form have been compared and it has been found that the additional parameter a controls the ordering between them. The distribution of the smallest and largest order statistic from the Exponentiated new Unit Lindley distribution has been derived and it has been found that both these densities are decreasing functions of the corresponding order statistics. The performance of the maximum likelihood estimates of the parameters have been studied through simulation study and it has been found that the bias and MSE of both a and θ decrease with an increase in the sample size. For showing the applicability of the proposed model, it has been applied to a real-life data set on timing of infant deaths in Uttar Pradesh for the time period 2005-06 and the fit has been compared with that of the latest version of the Unit Lindley distribution. It has been found that the Exponentiated version of the latest Unit Lindley distribution provides a better fit than the Unit Lindley distribution for the data set considered. Hence, it can be concluded that the proposed Exponentiated version of the latest Unit Lindley distribution is more flexible and can thus, serve as an alternative to the latest Unit Lindley distribution.

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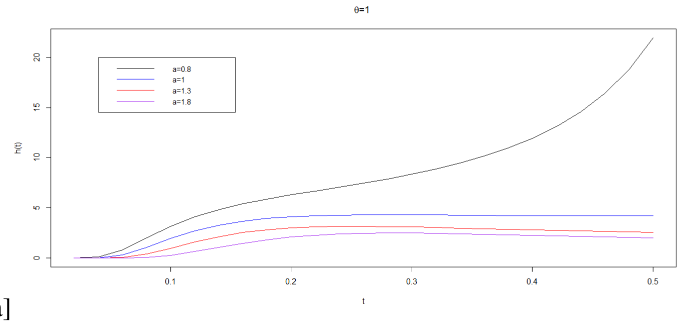

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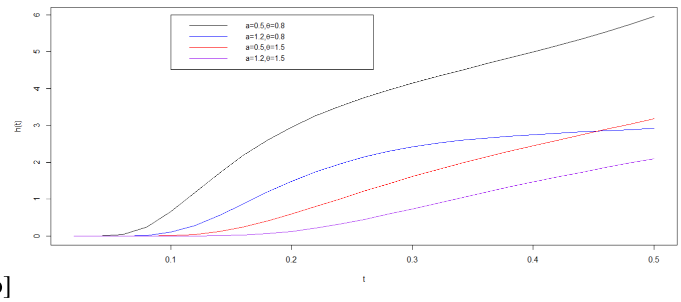
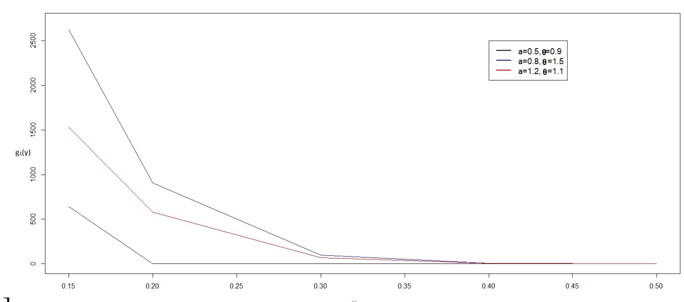
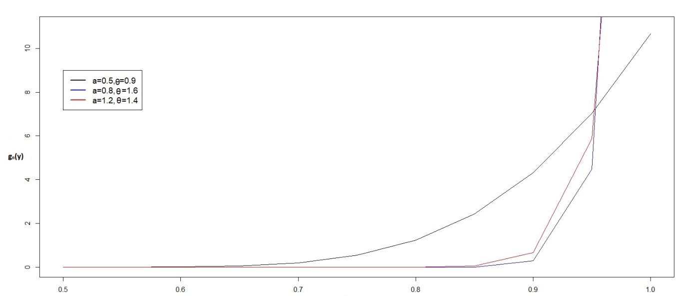


[b]

Figure 4. Hazard rate function plot for (a) different values of a (keeping θ fixed at 1) and (b) different values of θ and a



[a]



[b]

Figure 5. Density plots of (a) the smallest order statistic $y_{1:n}$ and (b) the largest order statistic $y_{n:n}$, for different values of a and θ and for fixed sample size $n = 50$