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Estimation of Age Replacement Policy when Maintenance Cost is Linear and Nonlinear

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Abstract: **The classical solution for replacement problem requires complete knowledge about the component life distribution. In this paper, suitable estimator of the age replacement strategy has been suggested and it is shown that the estimator is strongly consistent. A simulation study has been carried out to illustrate the efficiency of the estimator.**

Index Terms: **Age replacement policy, increase failure rate (IFR), kernel estimator, maintenance cost, strong consistency, rayleigh distribution.**

I. INTRODUCTION

Consider a system which is subject to failure due to failure of the components. The components can be replaced suitably and instantaneously in order to keep the system functional. Let the life distribution function of the component be $F : R^+ \to [0, 1)$, which is absolutely continuous having density function f and survival function $\bar{F} \equiv 1 - F$. Generally, a replacement policy involves routine replacement (on failure) and preventive replacement (in anticipation of failure). The mostly discussed replacement policies are the age replacement policy and the block replacement policy (cf. Arunkumar (1972), Ascher and Feingold (1984), Rigdon and Basu (2000)). In age replacement policy (see Bergman (1979), Frees and Ruppert (1985), Ingram and Scheaffer (1976), Jiang and Ji (2002), Lim, Qu and Zuo (2016), Park, Jung and Park (2015) and Yeh, Chen and Chen (2005)), a component is replaced by a new component of the same type on failure or after a specified age T, whichever is earlier. In periodic replacement policy, replacement is done by a new equipment of the same type at specified equidistant points of time $T, 2T, ...$ and only minimal repair is undertaken for any intervening failure in order to keep the system functional. By minimal repair we mean the system will be repaired in such a way that the failure rate of the system will remain same as that just prior to its failure. Roy and Basu (1993) have considered the estimation of age and periodic replacement policies, where they have taken only the replacement cost against failure and the planned replacement cost. Here we have considered the estimation of the age replacement policy taking maintenance cost of the system at different points of time into consideration, since it plays an important role in replacing the system. In most of the cases, it happens that after certain age of the system, though the system is functioning, its maintenance cost goes up in such a way that it becomes economical to replace the system.

Let $N(t)$ denote the number of shocks/failures occurring in $[0, t]$. We assume that $\{N(t), t \geq 0\}$ is a point process with $N(0) = 0$ and $N(t) < \infty$ for all $t \geq 0$ almost surely. Let τ be a stopping time with respect to the process $\{N(t), t \geq 0\}$ after which the system must be replaced. Further $N(t)$ is assumed to have sample paths with unit steps at points $\tau_1, \tau_2, ...$ with $0 < \tau_1 < \tau_2 < ...$ and $\tau_0 = 0$. We also assume that $N(t)$ and τ are independent. If the costs involved are due to incoming shocks, maintenance of the system, replacement of the system etc., it seems economical to replace the system at time $\min(t, \tau)$ for some optimally chosen t.

Define

$g(i, t)$ = cost per unit time of maintenance of the system at time $t \in [\tau_i, \tau_{i+1})$, $i = 1, 2, \ldots$

R_1 = Replacement cost of the system if it is replaced before failure.

R_2 = Replacement cost of the system if it is replaced at failure.

Then the average expected cost per unit time (cf. Barlow and Proschan (1996))is given by

$$C(t) = \frac{R_1 + \psi(t)}{\phi(t)},$$

where

$$\psi(t) = (R_2 - R_1).P(\tau \leq t) + \int_0^t E\left[g(N(u), u)I_{(\tau > u)}\right] du,$$
$$\phi(t) = E(t \wedge \tau) = \int_0^t P(\tau > u)du,$$

and $I_{(\cdot)}$ is the indicator function. If τ has distribution function F and density function f, then $\frac{d}{dt}C(t) = 0$ gives

$$\begin{aligned} L(T) &\stackrel{def}{=} \left[\int_0^T \bar{F}(u)du\right]\left[f(T) + E\left[g(N(T), T)\right]\bar{F}(T).\frac{\lambda}{R_1}\right] - \\ &\quad \left[\lambda + F(T) + \frac{\lambda}{R_1}\int_0^T E\left[g(N(u), u)\right]\bar{F}(u)du\right]\bar{F}(T) \\ &= 0, \end{aligned} \qquad (1)$$

where $\lambda = \frac{R_1}{R_2 - R_1}$ (> 0). Thus, the optimal age replacement time T_0 is given by $L(T_0) = 0$.

In Section 2 of this paper, we find T_n, an estimator of T_0 using kernel estimation method when the maintenance cost is linear and show that T_n is a strongly consistent estimator of T_0 if the underlying distribution is IFR (Increasing in Failure Rate). The same type of result is done in Section 3 when the maintenance cost is not linear. In Section 4 we present a simulation study to illustrate the workability of our estimation procedure.

II. ESTIMATION AND THE PROPERTIES OF THE ESTIMATOR WHEN MAINTENANCE COST IS PIECEWISE CONSTANT

Let $f(t)$ be the density function and $\bar{F}(t)$ be the survival function of the random variable X. Then the failure rate (or hazard rate) function $r_F(t)$ is defined as $r_F(t) = \frac{f(t)}{\bar{F}(t)}$, for all t such that $\bar{F}(t) > 0$. The random variable X is said to be IFR (Increasing in Failure Rate) if $r_F(t)$ is increasing in t (cf. Barlow, Marshall and Proschan (1963)).

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables having distribution function F, survival function $\bar{F}$, and $f_n(x)$ be a continuous *kernel estimator* of the uniformly continuous density function $f(x)$ corresponding to F. Consider a kernel function $K(x)$ such that

(i) $K(x)$ is uniformly continuous on $R = (-\infty, \infty)$.
(ii) $K(x)$ is of bounded variation on R.
(iii) There exists a ζ such that $K(x) = 0$ for $|x| > \zeta$.
(iv) $\int_{-\zeta}^{\zeta} K(x)dx = 1$.
(v) $\int_{-\zeta}^{\zeta} |K(x)|dx < \infty$.
(vi) $\int_{-\zeta}^{\zeta} |x \ln |x||^{1/2}|dK(x)| < \infty$.

Define $f_n(x)$ as

$$f_n(x) = \frac{1}{nb_n}\sum_{i=1}^{n} K\left[\frac{x - X_i}{b_n}\right],$$

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where $b_n(>0)$, the window width (or bandwidth) is taken as the n^{th} term of the sequence of positive numbers assumed to satisfy the following conditions (cf. Silverman (1978)).

(a) $b_n \to 0$ as $n \to \infty$.
(b) $\frac{nb_n}{\ln n} \to \infty$ as $n \to \infty$.

For further discussion on kernel density estimation, one can refer to Akaike (1954), Prakasa Rao (1983), Rosenblatt (1956), Schuster (1969), Van Ryzin (1969) to name a few.

Let us take $g(i,T) = \alpha + i\beta$, for $i = 1, 2, ...$ with $T \geq 0$ and $\{N(t),\ t \geq 0\}$, be a point process with $E[N(t)] = \delta t$. It is to be noted that we do not assume any specific stochastic process for $N(t)$. Here the maintenance cost is linear in i. For $T \in [\tau_i, \tau_{i+1})$, $g(i,T) = \alpha + i\beta$. This means, in each such interval, $g(i,T)$ is constant. That is, it is linear in i, but constant in T. In other words, the maintenance cost is a piecewise constant function. Then (1) reduces to

$$\begin{aligned} L(T) &= \left[\int_0^T \bar{F}(u)du\right]\left[f(T) + (\alpha + \beta\delta T)\bar{F}(T).\frac{\lambda}{R_1}\right] - \\ &\quad \left[\lambda + F(T) + \frac{\lambda}{R_1}\left(\alpha\int_0^T \bar{F}(u)du + \beta\delta\int_0^T u\bar{F}(u)du\right)\right]\bar{F}(T) \\ &= 0. \end{aligned} \tag{2}$$

Note that $T = \infty$ is always a solution to (2).

Define

$$L^*(T) = \frac{L(T)}{\bar{F}(T)}$$

If F is IFR, $L^*(T)$ is increasing in T and must have atmost one finite root.

Write

$$F_n(x) = 1 - \bar{F}_n(x) = \int_0^x f_n(t)dt$$

and

$$\begin{aligned} L_n(T) &= \left[\int_0^T \bar{F}_n(u)du\right]\left[f_n(T) + (\alpha + \beta\delta T)\bar{F}_n(T).\frac{\lambda}{R_1}\right] - \\ &\quad \left[\lambda + F_n(T) + \frac{\lambda}{R_1}\left(\alpha\int_0^T \bar{F}_n(u)du + \beta\delta\int_0^T u\bar{F}_n(u)du\right)\right] \\ &\quad .\bar{F}_n(T) = 0. \end{aligned} \tag{3}$$

Let

$$\begin{aligned} T_n &\stackrel{\text{def}}{=} \inf\{T : L_n(T) = 0\} \\ &= \min\{T : L_n(T) = 0\}. \end{aligned} \tag{4}$$

Theorem 1: If F is IFR having uniformly continuous density function $f(\cdot)$, then T_n, defined in (4), is a strongly consistent estimator of the optimal age replacement policy T_0.

Proof: Define

$$\delta_n = \sup\{|f_n(x) - f(x)| : x \geq 0\}. \tag{5}$$

Observe that

$$|F_n(x) - F(x)| \leq x\delta_n. \tag{6}$$

Case I: Let $T_0 < \infty$. First we show that the sequence $\{T_n\}$, defined in (4) is bounded above. For this, if possible, let $\{T_n\}$ be unbounded. So, for every positive number M,

$$T_n > M \text{ infinitely often (i.o.)}.$$

This further gives

$$L_n(M) \neq 0 \quad \text{i.o.}$$

But $L_n(0) = -\lambda(<0)$. So, $L_n(M) < 0$ i.o. for every $M > 0$, since $L_n(\cdot)$ is continuous. Thus we have, for every $M(>0)$, there exists an increasing sequence of indices $\{n_k\}$ such that for every k,

$\left[\int_0^M \bar{F}_{n_k}(u)du\right]\left[f_{n_k}(M) + (\alpha + \beta\delta M)\bar{F}_{n_k}(M).\frac{\lambda}{R_1}\right] -$
$\left[\lambda + F_{n_k}(M) + \frac{\lambda}{R_1}\left(\alpha\int_0^M \bar{F}_{n_k}(u)du + \beta\delta\int_0^M u\bar{F}_{n_k}(u)du\right)\right]\bar{F}_{n_k}(M)$
$< 0.$

Taking limit as $k \to \infty$, we have

$\left[\int_0^M \bar{F}(u)du\right]\left[f(M) + (\alpha + \beta\delta M)\bar{F}(M).\frac{\lambda}{R_1}\right] -$
$\left[\lambda + F(M) + \frac{\lambda}{R_1}\left(\alpha\int_0^M \bar{F}(u)du + \beta\delta\int_0^M u\bar{F}(u)du\right)\right]\bar{F}(M) < 0.$

Or, for every $M(>0)$,

$$\begin{aligned} \phi(M) &\stackrel{def}{=} \left[r(M) + (\alpha + \beta\delta M)\frac{\lambda}{R_1}\right]\int_0^M \bar{F}(u)du - \\ &\quad \left[\lambda + F(M) + \frac{\lambda}{R_1}\left(\alpha\int_0^M \bar{F}(u)du + \beta\delta\int_0^M u\bar{F}(u)du\right)\right] \\ &\quad \leq 0, \end{aligned} \tag{7}$$

where $r(x)$ is the failure rate corresponding to $F(x)$ at the point x. Taking derivative of the left-hand side of (7) with respect to M, we have

$$\begin{aligned} \phi'(M) &= \left[r'(M) + \frac{\beta\delta\lambda}{R_1}\right]\int_0^M \bar{F}(y)dy \\ &\geq 0, \end{aligned}$$

since F is IFR. If T_0 be a solution of (2), then $\phi(M) > 0$ for all $M > T_0$, which contradicts (7). Thus, there exists an $M(>0)$ such that $T_n \leq M$ for all n.

Now,

$$\begin{aligned} |L(T_n)| &= |L(T_n) - L_n(T_n)| \\ &\leq t_{1n} + t_{2n} + ... + t_{5n}, \end{aligned}$$

where

$$\begin{aligned} t_{1n} &= \left|f(T_n)\int_0^{T_n}\bar{F}(u)du - f_n(T_n)\int_0^{T_n}\bar{F}_n(u)du\right|, \\ t_{2n} &= \left|F(T_n)\bar{F}(T_n) - F_n(T_n)\bar{F}_n(T_n)\right|, \\ t_{3n} &= \lambda\left|F(T_n) - F_n(T_n)\right|, \\ t_{4n} &= \frac{\beta\delta\lambda T_n}{R_1}\left|\bar{F}(T_n)\int_0^{T_n}\bar{F}(u)du - \bar{F}_n(T_n)\int_0^{T_n}\bar{F}_n(u)du\right|, \end{aligned}$$

and

$$t_{5n} = \frac{\beta\delta\lambda}{R_1}\left|\bar{F}(T_n)\int_0^{T_n}u\bar{F}(u)du - \bar{F}_n(T_n)\int_0^{T_n}u\bar{F}_n(u)du\right|.$$

Note that

$$\begin{aligned} t_{1n} &= \left|\left[f(T_n)\int_0^{T_n}\bar{F}(u)du - f_n(T_n)\int_0^{T_n}\bar{F}(u)du\right]\right. \\ &\quad \left.+\left[f_n(T_n)\int_0^{T_n}\bar{F}(u)du - f_n(T_n)\int_0^{T_n}\bar{F}_n(u)du\right]\right| \\ &= \left|(f(T_n) - f_n(T_n))\int_0^{T_n}\bar{F}(u)du + \right. \\ &\quad \left. f_n(T_n)\int_0^{T_n}\left(\bar{F}(u) - \bar{F}_n(u)\right)du\right| \\ &\leq \delta_n\int_0^{T_n}\bar{F}(u)du + \delta_n f_n(T_n)\int_0^{T_n}udu \\ &\leq \delta_n\left[\mu + Mf_n(T_n)T_n\right] \\ &\leq \left[\mu + M\left(1 + \lambda + \frac{\lambda M^2\delta\beta}{R_1}\right)\right]\delta_n. \end{aligned}$$

The first inequality follows from (5) and (6), second inequality follows due to the fact that $\int_0^{T_n}\bar{F}(u)du \leq \mu = \int_0^\infty \bar{F}(u)du$ and $T_n \leq M$ for all n, whereas the last inequality can be obtained as under-

Since T_n is a solution of (3), we have

$$\begin{aligned} f_n(T_n)\int_0^{T_n}\bar{F}_n(u)du &= \bar{F}_n(T_n)\left[\lambda + F_n(T_n) + \right. \\ &\quad \left.\frac{\beta\delta\lambda}{R_1}\int_0^{T_n}u\bar{F}_n(u)du - \frac{\beta\delta\lambda T_n}{R_1}\int_0^{T_n}\bar{F}_n(u)du\right], \end{aligned}$$

which is equivalent to

$$\begin{aligned} f_n(T_n)T_n &\leq \lambda + F_n(T_n) + \frac{\beta\delta\lambda}{R_1}\int_0^{T_n}u\bar{F}_n(u)du - \\ &\quad \frac{\beta\delta\lambda T_n}{R_1}\int_0^{T_n}\bar{F}_n(u)du \\ &\leq 1 + \lambda + \frac{\beta\delta\lambda}{R_1}T_n^2 - \frac{\beta\delta\lambda}{R_1}\bar{F}_n(T_n)T_n^2 \\ &\leq 1 + \lambda + \frac{\beta\delta\lambda}{R_1}M^2. \end{aligned}$$

$$\begin{aligned}
t_{2n} &= \left|F(T_n) - F^2(T_n) - F_n(T_n) + F_n^2(T_n)\right| \\
&\leq |F(T_n) - F_n(T_n)| + \\
&\quad |F_n(T_n) + F(T_n)||F_n(T_n) - F(T_n)| \\
&\leq T_n\delta_n + 2T_n\delta_n \\
&\leq 3M\delta_n. \\
t_{3n} &\leq \lambda M\delta_n. \\
t_{4n} &\leq \frac{\beta\delta\lambda M}{R_1}\left[\left|\bar{F}(T_n) - \bar{F}_n(T_n)\right|\int_0^{T_n}\bar{F}(u)du + \right. \\
&\quad \left. \bar{F}_n(T_n)\int_0^{T_n}|\bar{F}(u) - \bar{F}_n(u)|du\right] \\
&\leq \frac{\beta\delta\lambda M}{R_1}\left[T_n\delta_n\int_0^{T_n}\bar{F}(u)du + \bar{F}_n(T_n)T_n\delta_n\right] \\
&\leq \frac{\beta\delta\lambda M^2}{R_1}\delta_n\left[\int_0^{T_n}\bar{F}(u)du + \bar{F}_n(T_n)\right] \\
&\leq M^2\delta_n\frac{\lambda\beta\delta}{R_1}(\mu + 1). \\
t_{5n} &\leq \frac{\lambda\beta\delta}{R_1}\left[\left|\bar{F}(T_n) - \bar{F}_n(T_n)\right|\int_0^{T_n}u\bar{F}(u)du + \right. \\
&\quad \left. \bar{F}_n(T_n)\int_0^{T_n}u|\bar{F}(u) - \bar{F}_n(u)|du\right] \\
&\leq \frac{\lambda\beta\delta}{R_1}\left[M\delta_n\int_0^{T_n}u\bar{F}(u)du + \delta_n\bar{F}_n(T_n)\int_0^{T_n}u^2du\right] \\
&\leq \frac{\lambda\beta\delta T_n}{R_1}\delta_n\left[M\int_0^{T_n}\bar{F}(u)du + \bar{F}_n(T_n)\frac{T_n^2}{3}\right] \\
&\leq M^2\delta_n\frac{\lambda\beta\delta}{R_1}\left(\mu + \frac{M}{3}\right).
\end{aligned}$$

Thus,

$$0 \leq |L(T_n)| \leq \theta\delta_n,$$

where $\theta < \infty$ and is independent of n. Hence, by Theorem A of Silverman (1978), we have

$$\lim_{n\to\infty}|L(T_n)| = 0 \quad a.e.$$

Now, if we take $\{T_{n_k}\}$, a convergent subsequence of $\{T_n\}$, the boundedness of $\{T_n\}$, together with the continuity of $L^*(\cdot)$, gives

$$L^*\left(\lim_{k\to\infty}T_{n_k}\right) = \lim_{k\to\infty}L^*\left(T_{n_k}\right) = \frac{\lim_{k\to\infty}L\left(T_{n_k}\right)}{\lim_{k\to\infty}\bar{F}\left(T_{n_k}\right)} = 0 = L^*\left(T_0\right).$$

Thus, T_0 being the unique root of $L^*(T) = 0$, we have

$$\lim_{k\to\infty}T_{n_k} = T_0 \quad a.e.$$

Further, as the limit is independent of the subsequence $\{T_{n_k}\}$, we have

$$P\left(\lim_{k\to\infty}T_{n_k} = T_0\right) = 1.$$

Case II: $T_0 = \infty$. If possible, let $\{T_n\}$ do not approach infinity as n tends to infinity. Then

$$T_n \leq M \quad \text{i.o. for some } M > 0.$$

So, we can find a convergent subsequence $\{T_{n_k}\}$ such that

$$T_{n_k} \to A < \infty.$$

Thus,

$$\begin{aligned}
L^*(A) &= L^*\left(\lim_{k\to\infty}T_{n_k}\right) \\
&= \lim_{k\to\infty}L^*\left(T_{n_k}\right) \\
&= \frac{\lim_{k\to\infty}L\left(T_{n_k}\right)}{\lim_{k\to\infty}\bar{F}\left(T_{n_k}\right)} \\
&= 0,
\end{aligned}$$

since L^* is continuous. This contradicts our hypothesis that $T_0 = \infty$. Hence the result.

III. ESTIMATION AND THE PROPERTIES OF THE ESTIMATOR WHEN MAINTENANCE COST IS NOT PIECEWISE CONSTANT

In practice maintenance cost need not increase linearly. To cope up this situation, in this section we analyze the case when the maintenance cost is a quadratic function. The general case when the maintenance cost is a higher order polynomial can be analyzed similarly.

Let us take $g(i,T) = \alpha + i\beta + i^2\gamma$, for $i = 1, 2, ...$ with $T \geq 0$ and $\{N(t),\ t \geq 0\}$, a point process with $E[N(t)] = \delta t$. Here α, β and γ are nonnegative because of the following reason:

Let $f(x) = a + bx + cx^2 > 0$ for all $x > 0$ and $f(x)$ be nondecreasing for all x. Then $f'(x) = 2cx + b \geq 0 \Rightarrow b \geq 0$ (otherwise, as $x \to 0, f'(x) < 0$). Further, $f(x) > 0$ for all $x > 0 \Rightarrow b^2 < 4ac \Rightarrow a$ and c will be of same sign. Again, $f(0) \geq 0 \Rightarrow a \geq 0$. Hence $a, b, c \geq 0$.

Now (1) reduces to

$$\begin{aligned}
L(T) &= \left[\int_0^T\bar{F}(u)du\right]\left[f(T) + \left\{\alpha + (\beta+\gamma)\delta T + \gamma(\delta T)^2\right\}\right. \\
&\quad \left. .\bar{F}(T).\frac{\lambda}{R_1}\right] - \left[\lambda + F(T) + \frac{\lambda}{R_1}\left(\alpha\int_0^T\bar{F}(u)du + \right.\right. \\
&\quad \left.\left. (\beta+\gamma)\delta\int_0^T u\bar{F}(u)du + \gamma\delta^2\int_0^T u^2\bar{F}(u)du\right)\right]\bar{F}(T) \\
&= 0. \qquad (8)
\end{aligned}$$

Define

$$L^*(T) = \frac{L(T)}{\bar{F}(T)}$$

Further, if F is IFR, $L^*(T)$ being an increasing function of T, must have atmost one finite root. Define

$$\begin{aligned}
L_n^*(T) &= \left[\int_0^T\bar{F}_n(u)du\right]\left[f_n(T) + \left\{\alpha + (\beta+\gamma)\delta T + \gamma(\delta T)^2\right\}\right. \\
&\quad \left. .\bar{F}_n(T).\frac{\lambda}{R_1}\right] - \left[\lambda + F_n(T) + \frac{\lambda}{R_1}\left(\alpha\int_0^T\bar{F}_n(u)du + \right.\right. \\
&\quad \left.\left. (\beta+\gamma)\delta\int_0^T u\bar{F}_n(u)du + \gamma\delta^2\int_0^T u^2\bar{F}_n(u)du\right)\right]\bar{F}_n(T).
\end{aligned}$$

Let

$$\begin{aligned}
T_n^* &\overset{def}{=} \inf\{T : L_n^*(T) = 0\} \\
&= \min\{T : L_n^*(T) = 0\}. \qquad (9)
\end{aligned}$$

The following theroem shows that the sequence $\{T_n^*\}$ is strongly consistent under certain condition. The proof is similar to that of *Theroem 1* with obvious modifications and is not given here.

Theorem 2: If F is IFR having a uniformly continuous density function $f(\cdot)$, then T_n^*, defined in (9), is a strongly consistent estimator of the optimal age replacement policy T_0.

Remark 1: One can show that the sequence $\{T_n^*\}$ is strongly consistent under the condition of the above theorem when the maintenance cost is a piecewise polynomial of degree m.

IV. A SIMULATION STUDY

We conclude our discussion with a simulation study to illustrate the efficiency of our estimation procedure compare to that proposed by Roy and Basu (1993) in the context of an age replacement policy. Since Rayleigh distribution has a wide range of applications in modeling life distributions of various electronic components, we take this distribution having survival function

$$\bar{F}(x) = e^{-\eta x^2};\ \eta > 0, x \geq 0,$$

as the underlying distribution. This is a strictly increasing failure rate (IFR) life distribution. For specific set of values of $\eta, \lambda, \alpha, \beta, \delta$ and R_1, we calculate the optimal age replacement time T_0 by solving equation (2). On the basis of the random observations generated from Rayleigh distribution, we calculate the Roy and Basu's estimator RB_n, as well as the estimator T_n as defined in (4). We then measure the absolute difference of T_0 from RB_n and also from T_n respectively. To increase the level of efficiency, the entire process is repeated 100 times. We note the proportion of occasions for which $|T_0 - T_n| < |T_0 - RB_n|$ and consider this proportion as a measure of efficiency of T_n over RB_n in estimating T_0. If this proportion exceeds 0.5, we conclude that T_n is better than RB_n for that set up.

We take, for simplicity, the following kernel function

$$K(x) = \frac{1}{2}, \text{if } |x| \leq 1$$

$$K(x) = 0, \text{otherwise},$$

and the window width b_n=$n^{-1/2}$. It is to be noted that the kernel estimator f_n of the underlying density function is piecewise continuous and vanishes outside [$X_{(min)}$-ζb_n,$X_{(max)}$+ζb_n]. We adopt an iterative method to solve the integral equations (2) and (3) and stop the iteration as soon as the difference of two consecutive solutions falls below 10^{-6}.

On the basis of 100 runs each involving 40 random observations, generated from the underlying distribution, the efficiency of the proposed estimator T_n is illustrated in Table 1. This shows that the estimator T_{40} (T_n based on $n = 40$) proposed in this paper is better than Roy and Basu's estimator RB_{40} for all the values of the parameters considered here. Column 7 of this table (E) indicate Eifficiency of T_n over RB_n.

Table I
EFFICIENCY OF T_n OVER RB_n

η	λ	α	β	δ	R_1	E	Better estimator
0.05	1	1	2	1	0.1	0.82	T_n
0.1	1	1	2	1	0.1	0.68	T_n
0.1	1	1	2	1	1.0	0.80	T_n
0.1	1	1	2	2	1.0	0.75	T_n
0.1	1	1	2	3	1.0	0.97	T_n
0.1	1	1	2	10	1.0	0.68	T_n
0.1	1	1	2	20	1.0	0.72	T_n
0.1	1	1	2	20	2.0	0.68	T_n
0.1	1	1	2	20	4.0	0.92	T_n
0.1	1	1	2	30	1.0	0.79	T_n
0.1	1	1	2	30	2.0	0.68	T_n
0.1	1	1	2	30	4.0	0.78	T_n
0.1	1	1	2	30	10	0.84	T_n
0.1	2	1	2	1	0.1	0.79	T_n
0.2	1	1	2	1	0.1	0.99	T_n
0.2	2	1	2	1	0.1	0.79	T_n
0.3	1	1	2	1	0.1	1.00	T_n
0.3	2	1	2	1	0.1	1.00	T_n
0.4	1	1	2	1	0.1	1.00	T_n
0.4	2	1	2	1	0.1	1.00	T_n
2.0	1	1	2	1	0.1	1.00	T_n
5.0	2	1	2	1	0.1	1.00	T_n

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