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The Kannan Contraction Mapping Theorem for Weighted Composition transformation in Metric Spaces

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***Abstract*—In this paper, fixed point theorems of the Kannan type are obtained in the setting of metric space and metric space endowed with partial order for weighted composition transformation.**

***Index Terms*—Weighted composition transformation, Kannan contraction mapping.**

I. INTRODUCTION

The Banach contraction principle (Banach, 1922) is one of the most useful tool in the theory of metric spaces (Abramovich & Aliprantis, 2002), (Conway, 1990). It has outstanding applications in Mathematics, Computer Science, Engineering and many other branches. Due to its importance and some of limitations, many authors has defined various other contractions mappings to meet out demands (Khan et al., 1984), (Rhoades, 2001). One of such contraction mapping is kannan contraction mapping. In 1968, the very first time R, Kannan has obtained fixed point results for a non continuous contraction mappings(Kannan, 1968). In 1975, P.V. Subrahmanyam has shown that Kannan mappings can be used in order to characterized the completeness of underlying space while even continuous contraction mapping does not guarantee to do so (Subrahmanyam, 1975). Recently, C.B. Ampadu in (Ampadu, 2020) has established Kannan Contraction Mapping Theorem for Composition transformation (Manhas, 1993) in Metric Spaces. The term Composition transformation was coined by Nordgren (Nordgren, 1968) in his paper entitled 'Composition Operators'. A detailed description of Composition transformation and Weighted Composition transformation is given in (Manhas, 1993). Earlier many authors investigated fixed point results for weighted composition operators, for more details see (Clahane, 2007) and (Vanani, 2018). In present paper, we have extended Ampadu (Ampadu, 2020) results for weighted composition transformation.

II. PRELIMINARIES

Now we present definitions and results useful for further discussion.

Definition 1: (Manhas, 1993) Let $\varphi : \mathbb{R} \to \mathbb{R}$ ba a self-map and u be a bounded function on $\mathbb{R}$. The weighted composition transformation uC_φ, induced by u and ϕ, is defined as

$$uC_\varphi(h) = u.(f \circ \varphi),$$

where, $\circ$ denotes composition of functions.

Definition 2: Let $(\mathbb{R}, d)$ be a metric space, and $uC_\varphi(h)$ be a self map on $(\mathbb{R}, d)$. We say $uC_\varphi(h)$ is a $u - \varphi - f -$ Kannan contraction if there exists if there exists $k \in [0, \frac{1}{2})$ such that

$$\begin{aligned} & d(uC_\varphi(h)(t), uC_\varphi(h)(s)) \\ \leq \quad & k[d(\varphi(t), uC_\varphi(h)(s)) + d(\varphi(s), uC_\varphi(h)(s))]. \end{aligned}$$

for all $t, s \in \mathbb{R}$.

Definition 3: (Manhas, 1993) Let $uC_\varphi(h)$ be a self map on $\mathbb{R}$, we say $\varphi(t) \in \varphi(\mathbb{R})$ is a fixed point of $uC_\varphi(h)$ if $uC_\varphi(h)(t) = \varphi(t)$.

Definition 4: Let $(\mathbb{R}, d)$ be a metric space. A sequence $\varphi(t_n) \in \mathbb{R}$ converges to $\varphi(t) \in \varphi(\mathbb{R})$ if for $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(\varphi(t_n), \varphi(t)) < \epsilon$$

whenever $n \geq n_0$.

Definition 5: (Abramovich & Aliprantis, 2002) Let $(\varphi(\mathbb{R}), d)$ be a metric space. A sequence $\varphi(t_n) \in \varphi(\mathbb{R})$ is called Cauchy sequence if for $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(\varphi(t_n), \varphi(t_m)) < \epsilon$$

whenever $m, n \geq n_0$.

Definition 6: (Abramovich & Aliprantis, 2002) Let $(\varphi(\mathbb{R}), d)$ be a metric space. It is said to be complete if every Cauchy sequence in $(\varphi(\mathbb{R}), d)$ is convergent in $(\varphi(\mathbb{R}), d)$.

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Definition 7: (Ampadu, 2020) Let $(X, \preceq)$ be a partially ordered set and $uC_\varphi(h)$ be a self map on X. Then $uC_\varphi(h)$ is non- decreasing if

$$uC_\varphi(h)(t_1) \preceq uC_\varphi(h)(t_2)$$

whenever $\varphi(t_2) \preceq \varphi(t_2)$ for all $\varphi(t_1), \varphi(t_2) \in \varphi(t)$.

Definition 8: Let S denote the class of functions $\beta : (0, \infty) \to [0, 1)$ with $\beta(t_n) \to 1$ implies $t_n \to 0$.

III. MAIN RESULTS

In the following theorem, we establish existence and uniqueness of fixed point for weighted composition transformation uC_φ.

Theorem 1: Let $(\mathbb{R}, d)$ be a metric space and $uC_\varphi(h)$ on $(\mathbb{R}, d)$ be $u-\varphi-f-$ Kannan contraction. The fixed point of $uC_\varphi(h)$ is unique provided $(\varphi(\mathbb{R}), d)$ is complete.

Proof: Define a sequence $\{\varphi(t_n)\}$ in $\varphi(\mathbb{R})$ by $\varphi(t_{n+1}) = uC_\varphi(h)(t_n)$. Then

$$\begin{aligned} & d(\varphi(t_{n+1}), \varphi(t_{n+2})) \\ = \;& d(uC_\varphi(h)(t_n), uC_\varphi(h)(t_n) \\ \leq \;& k[d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t_{n+1}), uC_\varphi(h)(t_n))] \\ \leq \;& k[d(\varphi(t_n), \varphi(t_{n+1})) + d(\varphi(t_{n+1}), \varphi(t_{n+2}))]. \end{aligned}$$

This asserts that

$$\begin{aligned} & d(\varphi(t_{n+1}), \varphi(t_{n+2})) \\ \leq \;& \frac{k}{1-k} d(\varphi(t_n), \varphi(t_{n+1})) \\ = \;& hd(\varphi(t_n), \varphi(t_{n+1})),\ where\ h = \frac{k}{1-k} \\ \leq \;& h^n d(\varphi(t_0), \varphi(t_1)). \end{aligned}$$

Now for $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} & d(\varphi(t_n), \varphi(t_m)) \\ \leq \;& d(\varphi(t_n), \varphi(t_{n+1})) + ... + d(\varphi(t_{m-1}), \varphi(t_m)) \\ \leq \;& h^n d(\varphi(t_0), \varphi(t_1)) + ... + h^{m-1} d(\varphi(t_0), \varphi(t_1)) \\ \leq \;& h^n d(\varphi(t_0), \varphi(t_1))(1 + h + h^2 + ...) \\ \leq \;& \frac{h^n}{1-h} d(\varphi(t_0), \varphi(t_1)). \end{aligned}$$

Hence,

$$\lim_{n\to\infty} d(\varphi(t_n), \varphi(t_m)) = 0$$

since $h < 1$. Therefore, sequence $\{\varphi(t_n)\}$ is a Cauchy sequence in $\varphi(\mathbb{R})$. Consequently, it is convergent in $\varphi(\mathbb{R})$ as $\varphi(\mathbb{R})$ is complete. Let $\varphi(t) \in \varphi(\mathbb{R})$ such that

$$\lim_{n\to\infty} (\varphi(t_n)) = \varphi(t).$$

We claim that, $\varphi(t) \in \varphi(\mathbb{R})$ is a fixed point of $uC_\varphi(h)(t)$. If possible, assume

$$uC_\varphi(h)(t) \neq \varphi(t).$$

That is

$$d(\varphi(t), uC_\varphi(h)(t)) > 0.$$

Consider,

$$\begin{aligned} & d(\varphi(t), uC_\varphi(h)(t)) \\ \leq \;& d(\varphi(t), uC_\varphi(h)(t_n)) + d(uC_\varphi(h)(t_n), uC_\varphi(h)(t)) \\ \leq \;& d(\varphi(t), \varphi(t_{n+1})) + k[d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t), uC_\varphi(h)(t))] \\ = \;& d(\varphi(t), \varphi(t_{n+1})) + k[d(\varphi(t_n), \varphi(t_{n+1})) + d(\varphi(t), uC_\varphi(h)(t))]. \end{aligned}$$

Taking $n \to \infty$, we get

$$d(\varphi(t), uC_\varphi(h)(t)) \leq kd(\varphi(t), uC_\varphi(h)(t)),$$

which is possible if

$$d(\varphi(t), uC_\varphi(h)(t)) = 0$$

since $1 - k > 0$ for any $k \in [0, \frac{1}{2})$. Hence, $\varphi(t) \in \varphi(\mathbb{R})$ is a fixed point of $C_\varphi(h)$. For uniqueness, if possible assume $\varphi(s)$ is other fixed point of $C_\varphi(h)$ such that $\varphi(t) \neq \varphi(s)$. Then

$$\begin{aligned} & d(\varphi(t), \varphi(s)) \\ = \;& d(uC_\varphi(h)(t), uC_\varphi(h)(s)) \\ \leq \;& k[d(\varphi(t), uC_\varphi(h)(t)) + d(\varphi(s), uC_\varphi(h)(s))] \\ \leq \;& k[d(\varphi(t), \varphi(t) + d(\varphi(s), \varphi(s))] \\ = \;& 0. \end{aligned}$$

That is $\varphi(t) = \varphi(s)$. Hence, fixed point is unique. ∎

Now, we obtain fixed point in case of partially ordered set.

Theorem 2: Let $(\mathbb{R}, \preceq)$ be a partially ordered set and $uC_\varphi(h)$ be a non- decreasing self map on $\mathbb{R}$ satisfying

$$\begin{aligned} & d(uC_\varphi(h)(t), uC_\varphi(h)(s)) \\ \leq \;& \beta\Big(\frac{d(\varphi(t), uC_\varphi(h)(t)) + d(\varphi(s), uC_\varphi(h)(s))}{2}\Big) \times \\ & \Big(\frac{d(\varphi(t), uC_\varphi(h)(t)) + d(\varphi(s), uC_\varphi(h)(s))}{2}\Big) \qquad (1) \end{aligned}$$

for all $\beta \in S$ and $\varphi(t), \varphi(s) \in \varphi(\mathbb{R}$ such that $\varphi(t) \preceq \varphi(s)$. Assume $(\varphi(\mathbb{R}), d)$ is complete and suppose further that either

1) $uC_\varphi(h)$ is continuous
2) $\varphi(\mathbb{R})$ has property, if a non-decreasing sequence $\varphi(t_n) \to \varphi(t)$, then $\varphi(t_n) \preceq \varphi(t)$ for all $n \geq 0$.

If there exists $\varphi(t) \in \varphi(\mathbb{R}$ such that $\varphi(t_0) \preceq uC_\varphi(h)(t_0)$, then $uC_\varphi(h)$ has a fixed point in $\varphi(\mathbb{R}$.

Proof: Let $\{\varphi(t_0)\}$ be such that $\varphi(t_0) \preceq uC_\varphi(h)(t_0)$. If $\varphi(t_0) \preceq uC_\varphi(h)(t_0)$, then $\varphi(t_0)$ is a fixed point. Assume, $\varphi(t_0) \prec uC_\varphi(h)(t_0)$ Define a sequence $\{\varphi(t_n)\}$ such that $\varphi(t_{n+1}) = uC_\varphi(h)(t_n)$. Since $uC_\varphi(h)$ is noon decreasing, so is $\{\varphi(t_n)\}$. If for some index m, $\varphi(t_{m+1}) = \varphi(t_m)\varphi(t_m)$ is a fixed point. Hence, assume, $\varphi(t_{n+1}) \prec \varphi(t_n)$. Then

$$\begin{aligned} & d(\varphi(t_{n+1}), \varphi(t_n)) \\ = \;& d(uC_\varphi(h)(t_n), uC_\varphi(h)(t_{n-1}) \\ \leq \;& \beta\Big(\frac{d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t_{n-1}), uC_\varphi(h)(t_{n-1}))}{2}\Big) \times \\ & \Big(\frac{d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t_{n-1}), uC_\varphi(h)(t_{n-1}))}{2}\Big) \\ \leq \;& \Big(\frac{d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t_{n-1}), uC_\varphi(h)(t_{n-1}))}{2}\Big) \\ = \;& \Big(\frac{d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t_{n-1}), uC_\varphi(h)(t_{n-1}))}{2}\Big). \end{aligned}$$

This asserts that

$$d(\varphi(t_{n+1}), \varphi(t_n)) \quad \leq \quad d(\varphi(t_{n-1}), \varphi(t_n)). \qquad (2)$$

Therefore, $\{\varphi(t_n)\}$ is a decreasing sequence non-negative real numbers. Consequently, it is convergent. Let r be the limit. Clearly, $r \geq 0$. We claim $r = 0$. If not,

$$\begin{aligned} &= \frac{1}{\frac{r}{r}} \\ &= \lim_{n\to\infty} \frac{d(\varphi(t_{n+1}), \varphi(t_n))}{d(\varphi(t_n), \varphi(t_{n-1}))} \\ &\leq \beta\Big(\frac{d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t_{n-1}), uC_\varphi(h)(t_{n-1}))}{2}\Big). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \beta\Big(\frac{d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t_{n-1}), uC_\varphi(h)(t_{n-1}))}{2}\Big) = 1.$$

Using definition 8, we get

$$\lim_{n\to\infty} \frac{d(\varphi(t_n), uC_\varphi(h)(t_n)) + d(\varphi(t_{n-1}), uC_\varphi(h)(t_{n-1}))}{2} = 0.$$

This implies that $r = 0$, a contradiction. Hence,

$$\lim_{n\to\infty} d(\varphi(t_n), \varphi(t_{n-1})) = 0. \qquad (3)$$

Again, we claim $\{\varphi(t_n)\}$ is a Cauchy sequence. If not, then for given $\epsilon > 0$ there exist two sub sequences $\{\varphi(t_{n(k)})\}, \{\varphi(t_{m(k)})\}$ such that $n(k) > m(k) > k$ and

$$d(\varphi(t_{m(k)}), \varphi(t_{n(k)})) \geq \epsilon. \qquad (4)$$

Now choose smallest $n(k)$ for each $k > 0$ corresponding to $m(k)$ such that inequality (4) holds. Then

$$d(\varphi(t_{m(k)}), \varphi(t_{n(k)-1})) < \epsilon.$$

Now for $k \geq 1$, we have

$$\begin{aligned} \epsilon &\leq d(\varphi(t_{m(k)}), \varphi(t_{n(k)})) \\ &\leq d(\varphi(t_{m(k)}), \varphi(t_{n(k)-1})) + d(\varphi(t_{n(k)-1}), \varphi(t_{n(k)})) \\ &< \epsilon + d(\varphi(t_{n(k)-1}), \varphi(t_{n(k)})). \end{aligned}$$

Using (3), we get

$$\lim_{k\to\infty} d(\varphi(t_{m(k)}, \varphi(t_{n(k)})) = \epsilon.$$

It is easy to see that $\varphi(t_{m(k)}$ and $\varphi(t_{n(k)})$ are comparable. Using the inequality (1), we get

$$\Big(\frac{1}{2} d(\varphi(t_{m(k)-1}), uC_\varphi(h)(t_{m(k)-1}))+ d(\varphi(t_{n(k)-1}), uC_\varphi(h)(t_{n(k)-1}))2\Big)$$

$$\begin{aligned} &\epsilon \\ &\leq d(\varphi(t_{m(k)}), \varphi(t_{n(k)})) \\ &\leq d(uC_\varphi(h)(t_{m(k)-1}), uC_\varphi(h)(t_{n(k)-1})) \\ &\leq \beta\Big(\frac{d(\varphi(t_{m(k)-1}), uC_\varphi(h)(t_{m(k)-1}))}{2} \\ &\quad + \frac{d(\varphi(t_{n(k)-1}), uC_\varphi(h)(t_{n(k)-1}))}{2}\Big) \\ &\times \Big(\frac{d(\varphi(t_{m(k)-1}), uC_\varphi(h)(t_{m(k)-1}))}{2} \\ &\quad + \frac{d(\varphi(t_{n(k)-1}), uC_\varphi(h)(t_{n(k)-1}))}{2}\Big) \\ &< \frac{d(\varphi(t_{m(k)-1}), uC_\varphi(h)(t_{m(k)-1}))}{2} \\ &\quad + \frac{d(\varphi(t_{n(k)-1}), uC_\varphi(h)(t_{n(k)-1}))}{2}. \end{aligned}$$

Again using the inequality (1), we get

$$\begin{aligned} &d(uC_\varphi(h)(s), \varphi(t_{n+1})) \\ &= d(uC_\varphi(h)(s), uC_\varphi(h)(t_n)) \\ &\leq \beta\Big(\frac{d(\varphi(s), uC_\varphi(h)(s) + d(\varphi(t_n), uC_\varphi(h)(t_n)}{2}\Big) \times \\ &\quad \Big(\frac{d(\varphi(s), uC_\varphi(h)(s)) + d(\varphi(t_n), uC_\varphi(h)(t_n))}{2}\Big) \\ &\leq \beta\Big(\frac{d(\varphi(s), uC_\varphi(h)(s)) + d(\varphi(t_n), (t_{n+1}))}{2}\Big) \times \\ &\quad \Big(\frac{d(\varphi(s), uC_\varphi(h)(s)) + d(\varphi(t_n), (t_n))}{2}\Big) \\ &< \frac{d(\varphi(s), uC_\varphi(h)(s)) + d(\varphi(t_n), \varphi(t_{n+1}))}{2}. \end{aligned}$$

However,

$$\lim_{n\to\infty} \varphi(t_n) = \varphi(s), \ and \ \lim_{n\to\infty} d(\varphi(t_{n+1}, \varphi(t_n)) = 0$$

asserts that

$$d(uC_\varphi(h)(s), \varphi(s) \leq \frac{d(\varphi(s), uC_\varphi(h)(s))}{2}.$$

Consequently,

$$C_\varphi(h)(s) = \varphi(s).$$

Hence, $C_\varphi(h)$ has the fixed point in $(\varphi(\mathbb{R}), d)$. ■

IV. Conclusion

We proved Kannan-type results in metric space and metric space endowed with partial order, respectively for self mappings that are weighted compositions transformations. Obtained assertions are generalization of consequences achieved by Ampadu Ampadu (2020).

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