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Type-I Progressive Hybrid Censoring Scheme: Bayesian Estimation for the Pareto Distribution of the Second Kind

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Abstract: **The use of Pareto Distribution of the second kind as a lifetime model is advocated by many authors. For this distribution, we derive maximum likelihood and Bayes estimators of the parameters and reliability function under type-I progressive hybrid censoring scheme. The Bayes estimates are obtained under squared-error, General Entropy (GE) loss function and LINEX loss functions using Lindley's approximation. A simulation study is also done to show the performance of the results.**

Index Terms: **Bayes estimator; GE loss function; LINEX loss function; Lindley's approximation; reliability function; squared-error loss function; type-I progressive hybrid censoring scheme.**

1. Introduction

Pareto (1897) introduced Pareto distribution as a model for the distribution of income. This model has several different forms studied by many authors including Davis and Feldstein (1979), Cohen and Whitten (1988), Grimshaw (1993). The Pareto distribution of the second kind also known as Lomax or Pearson's Type VI distribution [see Johnson et al. (1994)]. In addition the Pareto distribution has found application in the military arena. It has been found to provide a good model in biomedical problems, such as survival time following heart transplant [see Bain and Engelhardt (1992), Arnold (1983)]. Using the Pareto distribution, Dyer (1981) studied annual wage data of production line works in a large industrial firm Lomax (1954) used this distribution in the analysis of business failure data. The length of wire between flaws also follows a Pareto distribution [see Bain and Engelhardt (1992)]. Estimation, predictions and some inference concerning the Pareto distribution were discussed by Geisser (1984,1985), Arnold and Press (1989), Hossain and Zimmer (2000), Soliman (2001), Howlader and Hossain (2002), Madi and Raqab (2004) and Fernander (2006).

Let the random variable (rv) X follows Pareto distribution with probability density function (pdf) given by

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)},\quad x\geq 0,\alpha,\beta>0, \tag{1.1}$$

The parameter α is a shape parameter and β is scale parameter. The cumulative density function (cdf) of (1.1) can given as

$$F(x)=P[X\leq x]$$

$$=\int_0^t \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}dx$$

$$F(x)=1-\left(1+\frac{x}{\beta}\right)^{-\alpha} \tag{1.2}$$

Reliability function of (1.1) obtain as

$$R(t)=P(x>t)$$

$$=\left(1+\frac{t}{\beta}\right)^{-\alpha} \tag{1.3}$$

And Hazard rate of (1.1) can given as

$$h(t)=\lim_{\Delta t\to 0}\frac{P(t\leq T<t+\Delta t\mid T\geq t)}{\Delta t}$$

$$=\frac{f(t)}{R(t)}$$

$$h(t)=\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-1} \tag{1.4}$$

Hazard rate of Pareto distribution second kind is a decreasing function of time making it a suitable model for components which age with time.

Bayesian Estimation in reliability and life testing was introduced by Bhattacharya (1967). He considered the estimation of the reliability function for exponential distribution using squared error loss function (SELF) and type-II censoring scheme. Since then plenty of paper have been published in this field under the assumption of SELF. One may refer to the book Marts and Waller (1982) for some citations. However, the use of symmetric loss function may sometimes be impractical for estimating mean life or reliability function because of the recognition of the fact that over estimation is usually more serious than underestimation for such problems. Varian (1975) and Zellner (1986) discussed write a paper on Bayesian Estimation and Prediction Using Asymmetric Loss Functions. They considered LINEX loss function and proposed related estimation procedure. A useful asymmetric loss function educated by Calabria and Pulcini (1996) is general entropy loss functions (GELF). This loss is a generalization of the entropy loss used by several authors [see Day et al. (1987) and Dey and Liu (1992)].

In this paper, we consider the problem on type-I progressive Hybrid censoring of obtaining MLEs and Bayes estimators. In section 1, we consider Pareto Distribution of second kind and obtaining cdf, reliability function and hazard rate. In the section 2, we set-up for maximum likelihood estimation under type-I progressive censoring and obtain MLE using Newton Raphson Method. In section 3, using Lindley's approximation to solving ratio of integration we obtaining Bayes estimate of parameter and reliability function under square error loss function (SELF) General Entropy loss function (GELF) and LINEX loss function. In section 4 we perform simulation study and conclusion.

2. Maximum Likelihood estimation:

Suppose n units are placed on a life test denoted by X_1, X_2, …, X_n and R_1, …, R_m , m<n, is prefixed where $\sum_{i=1}^{m} R_i + m = n$. Suppose T is a pre-fixed time point. At the time of first failure $X_{1:m:n}$, R_1 is of the remaining n-1 units are randomly removed. Similarly at the time of second failure $X_{2:m:n}$, R_2 of the remaining n-R_1-2 surviving units is withdrawn and so on. When the m^th failure $X_{m:m:n}$ occurs before the time point T, the experiment stops at the time point $X_{m:m:n}$, when $X_{m:m:n}$ >T and only D failure occurs before the time point T, when $0 \le D < m$, then at time point T all the remaining R_D^* units is removed and we stop the experiment at the time point T. On observing above process we find two cases:

Case I: {$X_{1:m:n}$<, $X_{2:m:n}$ …<$X_{D:D:n}$} if $X_{m:m:n} > T$; type I progressive hybrid censoring scheme.

Case II: {$X_{1:m:n}$<…<$X_{m:m:n}$} if $X_{m:m:n} \le T$; Progressive type II censoring scheme.

where, D denotes the number of failures that occurs before time T.

Kundu and Joarder (2006) discussed this scheme and obtained MLE and Bayes estimators of the parameter of exponential distribution where as Childs et al. (2008) derived exact distribution of MLEs of parameter for the same distribution. Park et al. (2011) discussed fisher information in progressive hybrid censoring schemes. Balakrishnan and Kundu (2013) provided the detailed description of several variants of hybrid censoring, important results and their applications.

Let $X_{1:n}$ < $X_{2:n}$ < ...< $X_{D:n}$, if $X_{m:n}$ >T; where D denotes the number of failure that occurs before time T, then we terminate the experiment at T*=min(T, $X_{m:m:n}$). The likelihood function of the observed data *d* is

$$L(\Theta \mid d) = c\prod_{i=1}^{D} f(x_i,\Theta)R(x_i \mid \Theta)^{R_i}.R(T^* \mid \Theta)^{R^*} \tag{2.1}$$

whe

$$c = n(n-R_1-1)(n-R_1-R_2-2)...(n-R_1-R_2-...-R_{D-1}-D+1)$$

and if D=1, 2,..., m-1

where, $R_{D+1}^* = \sum_{k=D+1}^{m}(R_k+1)$

Θ is vector of parameter and using (1.1) and (1.3) we obtain from (2.1), that

$$L(\alpha,\beta;d) = c\prod_{i=1}^{D}\frac{\alpha}{\beta}\left(1+\frac{x_i}{\beta}\right)^{-(\alpha R_i+\alpha+1)}\left(1+\frac{T^*}{\beta}\right)^{-\alpha R^*} \tag{2.2}$$

Taking log both side of (2.2), we obtain log likelihood

$$l^* \propto D\log(\alpha)+D\log(\theta)-\sum_{i=1}^{D}x_i^{\alpha}+(\alpha-1)\sum_{i=1}^{D}\log(x_i)+(\theta-1)\sum_{i=1}^{D}\log(1-\exp(-x_i^{\alpha})) + \sum_{i=1}^{D}R_i\log(1-(1-\exp(-x_i^{\alpha}))^{\theta})+R^*\log(1-(1-\exp(-T^{*\alpha}))^{\theta}) \tag{2.3}$$

Partially differentiating (2.3) w.r.t. α, β we obtain

$$\frac{\partial l^*}{\partial\alpha} = \frac{m}{\alpha}-\sum_{i=1}^{m}\log\left(1+\frac{x_i}{\beta}\right)-\sum_{i=1}^{m}R_i\log\left(1+\frac{x_i}{\beta}\right)-R^*\log\left(1+\frac{T^*}{\beta}\right) \tag{2.4}$$

$$\frac{\partial l^*}{\partial\beta} = -\frac{D}{\beta}+\sum_{i=1}^{D}(1+\alpha)\frac{x_i}{\beta^2}\left(1+\frac{x_i}{\beta}\right)^{-1}+\sum_{i=1}^{D}\alpha\frac{R_i x_i}{\beta^2}\left(1+\frac{x_i}{\beta}\right)^{-1}+\alpha\frac{R^*T^*}{\beta^2}\left(1+\frac{T^*}{\beta}\right)^{-1} \tag{2.5}$$

The required estimates $\hat{\alpha}$ and $\hat{\beta}$ are to be found by solving simultaneously the two equations (2.4) and (2.5). Clearly these equations are transcendental equations in α and β no closed form solutions are known. It may be solved using an iterative numeric technique, such as Newton-Raphson iteration to get the estimates.

3. Bayesian Estimation Problem Using Lindley Approximation

We suppose some information on the shape parameters α and β is available priori. Formulation of joint prior density would be constructed. Let us consider a conditional prior distribution of β given α, which may appropriately be the conjugate gamma. The scale parameter of exponentiated Weibull distribution is α, which is assumed to become known previously with knowledge that may be translated into an exponential distribution the joint prior density function is

$$\pi(\alpha,\beta)=\pi_1(\alpha)\pi_2(\beta) \tag{3.1}$$

Let us consider gamma prior for α given by

$$\pi_1(\alpha)=\frac{\mu^{-\nu}\alpha^{\nu-1}}{\Gamma(\nu)}\exp\left(-\frac{\alpha}{\mu}\right),\ \mu>0,\upsilon>0, \tag{3.2}$$

and for β exponential prior

$$\pi_2(\beta)\propto 1/\beta, \tag{3.3}$$

the joint prior density of α and β comes out to be

$$\pi(\alpha,\beta)=\frac{\mu^{\upsilon}}{\beta\alpha^{\upsilon+1}\Gamma(\upsilon)}\exp\left(-\frac{\mu}{\alpha}\right);\qquad \alpha,\beta,\mu,\upsilon>0. \tag{3.4}$$

In what follows, we derive Bayes estimation of *α*, *β* and reliability function. Now the posterior expectation of any parametric function of $\omega=(\alpha,\beta)$ can be obtain by solving the following ratio of integrals.

$$E(\omega\,|\,d)=\frac{\int\omega\pi(\alpha,\beta)L(\alpha,\beta\,|\,d)d\alpha d\beta}{\int\pi(\alpha,\beta)L(\alpha,\beta\,|\,d)d\alpha d\beta} \tag{3.5}$$

This on Lindley's (1980) approximation can be written as follows.

$$E(\omega\,|\,d)=\omega+\frac{1}{2}[A+l_{30}B_{12}+l_{03}B_{21}+l_{21}C_{12}+l_{12}C_{21}]+\rho_1A_{12}+\rho_2A_{21} \tag{3.6}$$

where,

$$A=\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}\,\sigma_{ij},$$

$$l'_{\eta\xi}=\frac{\partial^{\eta+\xi}}{\partial\alpha^{\eta}\partial\beta^{\xi}},$$

$$\eta\,\&\,\xi=0,1,2,3;\ \eta+\xi=3,$$

for i, j=1, 2

$$\rho_i=\frac{\partial\rho}{\partial\lambda};\ \omega_{ij}=\frac{\partial^2\lambda}{\partial\alpha\partial\beta},$$

where,

$$\rho=\log\pi(\alpha,\beta)$$

and for $i\neq j$

$$A_{ij}=\omega_i\sigma_{ii}+\omega_j\sigma_{ij},\ B_{ij}=(\omega_i\sigma_{ii}+\omega_j\sigma_{ij})\sigma_{ii},\ C_{ij}=3\omega_i\sigma_{ii}\sigma_{ij}+\omega_j(\sigma_{ii}\sigma_{ij}+2\sigma_{ij}^2)$$

where, σ_{ij}, is the (i, j)th element in the inverse of the matrix $\{-l_{ij}\}$; $i,j=1,2,$ such that $l_{ij}=\frac{\partial^2 l}{\partial\alpha_i\partial\theta}$. For given expression of the likelihoods in form, we define the following notations.

Let

$$\sigma_{11}=\frac{H}{N},\ \sigma_{22}=\frac{G}{N},\ \sigma_{12}=\sigma_{21}=\frac{I}{N},\ where\ \ N=GH-I^2$$

.With these notations, can be written as follows [see Nassar & Essa (2004) and Kim, et al (2009)].

$$E(\omega\,|\,d)=\hat{\omega}+\omega_1\psi_1+\omega_2\psi_2+\phi \tag{3.7}$$

Where,

$$\psi_1=\frac{1}{N}(H\rho_1-I\rho_2)+\frac{1}{2N^2}[H^2l_{30}-IGl_{03}+(GH+2I^2)l_{12}-3IHl_{21}],$$

$$\psi_2=\frac{1}{N}(G\rho_2-I\rho_1)+\frac{1}{2N^2}[G^2l_{03}-IGl_{30}+(GH+2I^2)l_{21}-3IGl_{12}],$$

and

$$\phi=\frac{1}{2N}[H\omega_{11}-I(\omega_{12}+\omega_{21})+G\omega_{22}]$$

,

for an estimation problem, we have

$$G=\frac{-\partial^2 l^*}{\partial\alpha^2},\qquad H=\frac{-\partial^2 l^*}{\partial\beta^2},\qquad I=\frac{\partial^2 l^*}{\partial\alpha\partial\beta}$$

where,

$$R^*_{D+1}=\sum_{k=D+1}^{m}(R_k+1).$$

Remark 1: Here we see that ψ_1 and ψ_2 are independent from ω and its derivatives i.e. ω_i's and ω_{ij}'s. Thus, once a Bayesian estimation problem is set in the above form, the posterior expectation of any parametric function ω can be obtained by calculating φ only. Thus for our estimation problem, we have.

3.1 Bayesian Estimation under SELF

Using the fact that, the Bayes estimator of any parameter is its posterior mean, we obtain the Bayes estimation of α, θ, Reliability function *R(t)* by using (3.7) as follows.

1. If $\omega = \alpha$, we have $\omega_1 = 1, \omega_2 = 0$ and $\varphi = 0$ substituting these values in (3.7), we get the following Bayes estimates of α under SELF
$$\tilde{\alpha}_s = \hat{\alpha} + \psi_1 \tag{3.8}$$
2. If $\omega = \beta$, we have $\omega_1 = 0, \omega_2 = 1$ and $\varphi = 0$ substituting these values in (3.7), we get the Bayes estimates of θ given by
$$\tilde{\beta}_s = \tilde{\beta} + \psi_2 \tag{3.9}$$
3. If $\omega = R(t) = \left(1 + \frac{t}{\beta}\right)^{-\alpha}$ we have
$$R_s(t) = \hat{R}(t) + \phi_R + \omega_1\psi_1 + \omega_2\psi_2, \tag{3.10}$$

3.2 Bayesian Estimation under LINEX loss Function

The LINEX loss function for estimating a parameter (parametric function) ω by $\tilde{\omega}$ proposed by calebria and pulcini (1996) is given by

$$L(\hat{\omega} - \omega) = a\exp(c(\hat{\omega} - \omega)) - c(\hat{\omega} - \omega) - 1, \quad a > 0,\ c \neq 0 \tag{3.11}$$

The Bayes Estimate of α under LINEX loss function is

$$\tilde{\omega}_l = -\frac{1}{c}\log[E_\omega(\exp(-c\omega))], \tag{3.12}$$

Provided $E_\omega(\exp(-c\omega))$ exist and finite. Thus to obtain $\tilde{\omega}$ the Bayes estimate of ω under LINEX loss function, we first find the posterior expectation $E_\omega(\exp(-c\omega))$ for given *c*, using (1) and (2) then the Bayes estimates of α, β and R(t) can be obtained using (3.7) and (3.12).

1. When $\omega = \exp(-c\alpha)$, then we have
$$\tilde{\alpha}_l = -\frac{1}{c}\log E_\alpha[\exp(-c\hat{\alpha})]$$
$$= -\frac{1}{c}\log[\exp(-c\hat{\alpha}) + \phi''_\alpha + \omega_1\psi_1] \tag{3.13}$$
where,
$$\omega_1 = -c\exp(-c\alpha)$$
$$\phi''_\alpha = \frac{1}{2N}[Gc^2\exp(-c\alpha)]$$
2. When $\omega = \exp(-c\beta)$, then we have
$$\tilde{\beta}_l = -\frac{1}{c}\log E_\beta[\exp(-c\hat{\beta})]$$
$$= -\frac{1}{c}\log[\exp(-c\hat{\beta}) + \phi''_\beta + \omega_2\psi_2] \tag{3.14}$$
Where
$$\omega_2 = -c\exp(-c\beta)$$
$$\phi''_\beta = \frac{1}{2N}[Hc^2\exp(-c\beta)]$$
3. When
$$\omega = R_l(t) = \exp(-cR(t)) = \exp\left(-c\left(1 + \frac{t}{\beta}\right)^{-\alpha}\right),$$
then we have
$$\tilde{R}_l = -\frac{1}{c}\log E_R\left[\exp\left(-c\left(1 + \frac{t}{\beta}\right)^{-\alpha}\right)\right]$$
$$= -\frac{1}{c}\log\left[\exp\left(-c\left(1 + \frac{t}{\beta}\right)^{-\alpha}\right) + \phi''_R + \omega_1\psi_1 + \omega_2\psi_2\right] \tag{3.15}$$

3.3 Bayesian Estimation under GELF

The General entropy loss function (GELF) for estimating a parameter (parametric function) ω by $\hat{\omega}$ proposed by calebria and pulcini (2006) is given by

$$L(\hat{\omega}, \omega) = \left(\frac{\hat{\omega}}{\omega}\right)^q - q\log\left(\frac{\hat{\omega}}{\omega}\right) - 1, \quad q \neq 0 \tag{3.16}$$

The Bayes Estimate of ω under GELF is

$$\tilde{\omega}_G = [E_\omega(\omega^{-q})]^{-1/q}, \tag{3.17}$$

Provided $E_\omega(\omega^{-q})$ exist and finite. Thus to obtain $\tilde{\omega}_G$ the Bayes estimate of ω under GELF, we first find the posterior expectation $E_\omega(\omega^{-q})$ for given q, using (3.7), the *n* the Bayes estimators of α, β and R(t) under GELF, can be obtained using (3.7).

1. When $\omega = \alpha^{-q}$, then using (3.7) we have
$$\tilde{\alpha} = (\hat{\alpha}^{-q} + \phi'_\alpha + \omega_1\psi_1)^{-1/q} \tag{3.18}$$
Where $\omega_1 = -q\hat{\alpha}^{-q-1}$
$$\phi'_\alpha = \frac{(q + q^2)H\alpha^{-q-2}}{2N}$$

2. When $\omega = \beta^{-q}$, then we have

$$\tilde{\beta} = (\hat{\beta}^{-q} + \phi'_{\beta} + \omega_2 \psi_2)^{-1/q} \tag{3.19}$$

Where $\omega_2 = -q\beta^{-q-1}$

$$\phi'_{\beta} = \frac{(q+q^2)G\beta^{-q-2}}{2N}$$

3. When $\omega = R_G(t)^{-q} = \left(1 + \frac{t}{\beta}\right)^{q\alpha}$, then we have

$$\tilde{R}_G(t) = (\hat{R}_G(t)^{-q} + \phi'_R + \omega_1\psi_1 + \omega_2\psi_2)^{-1/q} \tag{3.20}$$

4. Simulation Study

In this section we present simulation study to show how one can apply the derived results for data analysis. We considered Pareto distribution of second kind. For the values of the parameters α=1.2 and β=0.5. we generate the type-I progressive hybrid censoring sample using the algorithm of Balakrishnan and Sandu (1995) in software *R*. Since the evaluation of Bayes estimators through Lindley' approximation requires the MLEs of parameters, we first obtain the MLEs using (2.4), (2.5). For the values of prior Hyper-parameters $\mu = 2$ and $\upsilon = 2$, we consider different progressive censoring at different level of sample size schemes and evaluate the values of Bayes estimators of both the parameters as well as their respective root mean squared errors(RMSEs). Here the samples are generated with the values of parameters to be α=1.2 and β=0.5.

In this paper we have shown that the computations for the Bayes estimators become easy with the present form of Lindley's approximation. Once the Bayesian estimation problem is set according to Section 3, we can compute the posterior expectation of any parametric function.

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Table 1: Estimate & RMSE in Parenthesis of α for α=1.2, β=0.5, μ=2, υ=2, sample=2000.

Sample Size		Progressive Censoring Scheme	$\hat{\alpha}$	$\tilde{\alpha}_S$	$\tilde{\alpha}_l$		$\tilde{\alpha}_q$	
n	m				c=-2	c=2	q=-2	q=2
100	100	Complete	1.7382	2.1286	1.9783	1.8658	1.9946	1.9505
			(1.5497)	(1.5492)	(0.8626)	(0.7176)	(1.1175)	(1.4657)
T	80	20*1,0*40,0*39	1.7169	2.2441	1.9574	1.8480	1.9980	1.8599
=			(0.5331)	(1.8798)	(0.8517)	(0.7186)	(1.2789)	(0.8128)
$X_{m:m:n}$		0*40,2*10, 0*30	1.0880	1.0972	1.1148	1.0800	1.1051	1.0732
			(0.1608)	(0.1530)	(0.1457)	(0.1625)	(0.1486)	(0.1678)
		0*30,1*20, 0*30	1.5956	1.9026	1.7994	1.7033	1.8050	1.7693
			(0.4100)	(1.8959)	(0.6825)	(0.5665)	(0.8768)	(1.1673)
T	80	20*1,0*40,0*39	2.7143	2.6310	2.6769	2.4040	2.6687	2.5361
=			(1.5326)	(1.4450)	(1.4866)	(1.2134)	(1.4831)	(1.3499)
0.75		0*40,2*10, 0*30	1.1960	1.2158	1.2441	1.1926	1.2266	1.1826
			(0.1247)	(0.1243)	(0.1380)	(0.1196)	(0.1275)	(0.1207)
		0*30,1*20, 0*30	1.8339	1.8505	1.9349	1.7809	1.8728	1.7842
			(0.6689)	(0.6803)	(0.7648)	(0.6070)	(0.7026)	(0.6151)
T	80	20*1,0*40,0*39	2.5700	2.4943	2.5239	2.3014	2.5254	2.4150
=			(1.3881)	(1.3085)	(1.3329)	(1.1112)	(1.3399)	(1.2291)
1		0*40,2*10, 0*30	1.1412	1.1566	1.1791	1.1369	1.1659	1.1284
			(0.1221)	(0.1149)	(0.1140)	(0.1211)	(0.1128)	(0.1256)
		0*30,1*20, 0*30	1.9122	1.9126	1.9891	1.8336	1.9345	1.8499
			(0.7347)	(0.7319)	(0.8078)	(0.6498)	(0.7538)	(0.6696)

Note: 7*1,0*12,0*5 indicate 7,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0

Table 2: Estimate & RMSE in Parenthesis of β for α=1.2, β=0.5, μ=2, υ=2, sample=2000.

Sample size n	m	Progressive Censoring Scheme	$\hat{\beta}$	$\tilde{\beta}_S$	$\tilde{\beta}_l$ c=-2	$\tilde{\beta}_l$ c=2	$\tilde{\beta}_q$ q=-2	$\tilde{\beta}_q$ q=2
100	100	Complete	0.9858	0.9472	0.9310	0.9446	0.9451	0.9487
			(0.5483)	(0.4908)	(0.4613)	(0.4855)	(0.4865)	(0.4969)
T	80	20*1,0*40,0*39	0.9516	0.9135	0.8953	0.9112	0.9124	0.9140
=			(0.5379)	(0.4714)	(0.4292)	(0.4674)	(0.4660)	(0.4794)
$X_{m:m:n}$		0*40,2*10, 0*30	0.4388	0.4353	0.4383	0.4361	0.4365	0.4319
			(0.1121)	(0.1131)	(0.1121)	(0.1124)	(0.1128)	(0.1143)
		0*30,1*20, 0*30	0.8299	0.7991	0.7895	0.7993	0.7988	0.7985
			(0.3972)	(0.3507)	(0.3281)	(0.3473)	(0.3486)	(0.3539)
T	80	20*1,0*40,0*39	0.5518	0.5425	0.5516	0.5418	0.5467	0.5312
=			(0.1266)	(0.1229)	(0.1280)	(0.1195)	(0.1256)	(0.1158)
0.75		0*40,2*10, 0*30	0.3631	0.3557	0.3613	0.3591	0.3571	0.3519
			(0.1566)	(0.1621)	(0.1577)	(0.1591)	(0.1610)	(0.1649)
		0*30,1*20, 0*30	0.4567	0.4433	0.4532	0.4476	0.4462	0.4359
			(0.1002)	(0.1045)	(0.1008)	(0.1013)	(0.1037)	(0.1070)
T	80	20*1,0*40,0*39	0.6168	0.6091	0.6173	0.6070	0.6131	0.5982
=			(0.1751)	(0.1702)	(0.1770)	(0.1660)	(0.1737)	(0.1605)
1		0*40,2*10, 0*30	0.3995	0.3929	0.3978	0.3956	0.3942	0.3893
			(0.1305)	(0.1344)	(0.1313)	(0.1323)	(0.1336)	(0.1366)
		0*30,1*20, 0*30	0.5298	0.5178	0.5268	0.5199	0.5209	0.5096
			(0.1069)	(0.1022)	(0.1054)	(0.1007)	(0.1036)	(0.0988)

Table 3: Estimate & RMSE in Parenthesis of Reliability for α=1.2, β=0.5, μ=2, υ=2, sample=2000, t=0.5.

Sample Size n	m	Progressive Censoring Scheme	$\hat{R}(t)$	$\tilde{R}_S(t)$	$\tilde{R}_l(t)$ c=-2	$\tilde{R}_l(t)$ c=2	$\tilde{R}_q(t)$ q=-2	$\tilde{R}_q(t)$ q=2
100	100	Complete	0.4762	0.3741	0.4230	0.2438	0.4355	0.4701
			(0.0740)	(0.6223)	(0.1913)	(0.3565)	(0.0681)	(0.0679)
T	80	20*1,0*40,0*39	0.4345	0.4335	0.4355	0.3950	0.4299	0.4340
=			(0.0430)	(0.0416)	(0.0413)	(0.0536)	(0.0425)	(0.0428)
$X_{m:m:n}$		0*40,2*10, 0*30	0.4640	0.3977	0.4164	0.2547	0.4267	0.4595
			(0.0646)	(0.3078)	(0.1188)	(0.3140)	(0.0647)	(0.0591)
		0*30,1*20, 0*30	0.1718	0.1954	0.2249	0.1222	0.1819	0.1717
			(0.2647)	(0.2415)	(0.2118)	(0.3149)	(0.2548)	(0.2647)
T	80	20*1,0*40,0*39	0.3530	0.3497	0.3532	0.2937	0.3427	0.3526
=			(0.0847)	(0.0876)	(0.0841)	(0.1420)	(0.0943)	(0.0851)
0.75		0*40,2*10, 0*30	0.2569	0.2609	0.2737	0.1911	0.2479	0.2568
			(0.1816)	(0.1771)	(0.1642)	(0.2457)	(0.1899)	(0.1818)
		0*30,1*20, 0*30	0.2150	0.2356	0.2536	0.1873	0.2249	0.2150
			(0.2222)	(0.2022)	(0.1841)	(0.2509)	(0.2127)	(0.2222)
T	80	20*1,0*40,0*39	0.3935	0.3900	0.3927	0.3391	0.3850	0.3930

=			(0.0471)	(0.0497)	(0.0474)	(0.0972)	(0.0542)	(0.0474)
1		0*40,2*10, 0*30	0.2789	0.2849	0.2953	0.2281	0.2741	0.2787
			(0.1602)	(0.1539)	(0.1434)	(0.2095)	(0.1644)	(0.1603)
		0*30,1*20, 0*30	0.4762	0.3741	0.4230	0.2438	0.4355	0.4701
			(0.0740)	(0.6223)	(0.1913)	(0.3565)	(0.0681)	(0.0679)

Table 4: Estimate & RMSE in Parenthesis of α for α=1.2, β=0.5, μ=2, υ=2, sample=2000.

Sample Size		Progressive Censoring Scheme	$\hat{\alpha}$	$\tilde{\alpha}_S$	$\tilde{\alpha}_l$		$\tilde{\alpha}_q$	
n	m				c=-2	c=2	q=-2	q=2
25	25	Complete	1.6119	2.0602	1.8668	1.6888	1.8645	1.7157
			(0.4811)	(1.0403)	(0.7081)	(0.5493)	(0.9776)	(0.7957)
T	18	7*1,0*12,0*5	1.5971	1.9837	1.8396	1.6207	1.8361	1.6706
=			(0.5255)	(1.3052)	(0.6812)	(0.4862)	(0.8647)	(0.8939)
$X_{m:m:n}$		0*12,0*5,7*1	1.0365	1.1053	1.1704	1.0226	1.1386	0.9942
			(0.3074)	(0.2716)	(0.3069)	(0.2823)	(0.2754)	(0.3013)
		0*6,0*5,1*7	1.0922	1.1618	1.2331	1.0745	1.1962	1.0490
			(0.2759)	(0.2396)	(0.2905)	(0.2369)	(0.2510)	(0.2557)
		0*6,1*7,0*5	1.3667	1.5919	1.5957	1.3941	1.5605	1.4081
			(0.3188)	(1.2554)	(0.4665)	(0.2655)	(0.5041)	(0.4943)
T	18	7*1,0*12,0*5	2.2471	2.0048	2.1503	1.7227	2.1015	1.8441
=			(1.1139)	(0.8293)	(0.9823)	(0.5410)	(0.9250)	(0.6744)
1.5		0*12,0*5,7*1	0.8322	0.9121	0.9375	0.8490	0.9350	0.8244
			(0.4006)	(0.3321)	(0.3248)	(0.3827)	(0.3161)	(0.4024)
		0*6,0*5,1*7	1.0615	1.1342	1.2012	1.0518	1.1680	1.0199
			(0.2440)	(0.2053)	(0.2327)	(0.2261)	(0.2070)	(0.2478)
		0*6,1*7,0*5	1.5898	1.6119	1.7855	1.4316	1.6775	1.4339
			(0.4769)	(0.4634)	(0.6420)	(0.2788)	(0.5297)	(0.3034)
T	18	7*1,0*12,0*5	2.0740	1.9138	2.0693	1.6583	1.9997	1.7531
=			(0.9368)	(0.7425)	(0.8947)	(0.4774)	(0.8254)	(0.5858)
2		0*12,0*5,7*1	0.7500	0.8293	0.8408	0.7714	0.8482	0.7532
			(0.4721)	(0.4011)	(0.3993)	(0.4522)	(0.3859)	(0.4665)
		0*6,0*5,1*7	0.9948	1.0689	1.1221	0.9964	1.0986	0.9652
			(0.2646)	(0.2120)	(0.2124)	(0.2528)	(0.2018)	(0.2764)
		7*1,0*12,0*5	2.0771	1.9171	2.0674	1.6593	1.9986	1.7548
			(0.9404)	(0.7407)	(0.8974)	(0.4786)	(0.8249)	(0.5880)

Table 5: Estimate & RMSE in Parenthesis of β for α=1.2, β=0.5, μ=2, υ=2, sample=2000.

Sample Size		Progressive Censoring Scheme	$\hat{\beta}$	$\tilde{\beta}_S$	$\tilde{\beta}_l$		$\tilde{\beta}_q$	
n	m				c=-2	c=2	q=-2	q=2
25	25	Complete	0.9131	0.8554	0.8451	0.8478	0.8575	0.8479
			(0.5634)	(0.4759)	(0.4454)	(0.4647)	(0.4746)	(0.4775)
T	18	7*1,0*12,0*5	0.9101	0.8426	0.8323	0.8308	0.8479	0.8299
=			(0.6056)	(0.5035)	(0.4739)	(0.4870)	(0.5079)	(0.5002)
$X_{m:m:n}$		0*12,0*5,7*1	0.4415	0.4074	0.4304	0.4159	0.4135	0.3952
			(0.2115)	(0.2059)	(0.2049)	(0.1956)	(0.2070)	(0.2049)

		0*6,0*5,1*7	0.4674	0.4343	0.4562	0.4417	0.4403	0.4220
			(0.2009)	(0.1922)	(0.1940)	(0.1828)	(0.1940)	(0.1895)
		0*6,1*7,0*5	0.6785	0.6302	0.6447	0.6303	0.6361	0.6173
			(0.3408)	(0.2889)	(0.2848)	(0.2743)	(0.2933)	(0.2795)
T	18	7*1,0*12,0*5	0.8167	0.7917	0.8372	0.7571	0.8101	0.7441
=			(0.4771)	(0.4961)	(0.5443)	(0.4194)	(0.5134)	(0.4324)
1.5		0*12,0*5,7*1	0.5047	0.4674	0.4886	0.4739	0.4730	0.4560
			(0.2236)	(0.2016)	(0.2064)	(0.1912)	(0.2048)	(0.1967)
		0*6,0*5,1*7	0.4858	0.4488	0.4723	0.4556	0.4552	0.4357
			(0.2211)	(0.2062)	(0.2116)	(0.1954)	(0.2092)	(0.2014)
		0*6,1*7,0*5	0.6176	0.5698	0.6040	0.5683	0.5816	0.5458
			(0.2785)	(0.2486)	(0.2732)	(0.2274)	(0.2595)	(0.2262)
T	18	7*1,0*12,0*5	0.8418	0.8123	0.8514	0.7750	0.8279	0.7716
=			(0.5129)	(0.5334)	(0.5671)	(0.4302)	(0.5437)	(0.4768)
2		0*12,0*5,7*1	0.5160	0.4808	0.4981	0.4851	0.4857	0.4705
			(0.2519)	(0.2237)	(0.2259)	(0.2120)	(0.2270)	(0.2184)
		0*6,0*5,1*7	0.4862	0.4519	0.4726	0.4581	0.4575	0.4401
			(0.2227)	(0.2059)	(0.2093)	(0.1964)	(0.2085)	(0.2020)
		0*6,1*7,0*5	0.8464	0.8148	0.8560	0.7843	0.8311	0.7748
			(0.5114)	(0.5068)	(0.5584)	(0.4468)	(0.5249)	(0.4567)

Table 6: Estimate & RMSE in Parenthesis of Reliability for α=1.2, β=0.5, μ=2, υ=2, sample=2000, t=0.5.

Sample Size		Progressive Censoring Scheme	$\hat{R}(t)$	$\widetilde{R}_S(t)$	$\widetilde{R}_l(t)$		$\widetilde{R}_q(t)$	
n	m				c=-2	c=2	q=-2	q=2
25	25	Complete	0.4776	0.3759	0.4244	0.1781	0.4284	0.4687
			(0.1093)	(0.9778)	(0.1514)	(0.3895)	(0.0917)	(0.0993)
T	18	7*1,0*12,0*5	0.4764	0.3755	0.4325	0.1552	0.4253	0.4657
=			(0.1205)	(0.9458)	(0.1313)	(0.4227)	(0.0983)	(0.1082)
$X_{m:m:n}$		0*12,0*5,7*1	0.4463	0.4254	0.4351	0.1918	0.4008	0.4424
			(0.0826)	(0.0722)	(0.0687)	(0.2466)	(0.0858)	(0.0799)
		0*6,0*5,1*7	0.4430	0.4221	0.4323	0.1976	0.4013	0.4393
			(0.0826)	(0.0714)	(0.0675)	(0.2417)	(0.0834)	(0.0800)
		0*6,1*7,0*5	0.4553	0.3851	0.4139	0.1424	0.4017	0.4479
			(0.1033)	(0.4266)	(0.1381)	(0.3706)	(0.0943)	(0.0958)
T	18	7*1,0*12,0*5	0.3263	0.3850	0.4123	0.3042	0.3523	0.3259
=			(0.1285)	(0.1091)	(0.0908)	(0.2585)	(0.1213)	(0.1289)
1.5		0*12,0*5,7*1	0.5521	0.5151	0.5202	0.2106	0.5010	0.5447
			(0.1238)	(0.0879)	(0.0927)	(0.2270)	(0.0745)	(0.1161)
		0*6,0*5,1*7	0.4580	0.4338	0.4426	0.2011	0.4130	0.4539
			(0.0694)	(0.0578)	(0.0581)	(0.2501)	(0.0621)	(0.0661)
		0*6,1*7,0*5	0.3772	0.3795	0.3997	0.1906	0.3484	0.3751
			(0.0915)	(0.0870)	(0.0740)	(0.2614)	(0.1090)	(0.0918)
T	18	7*1,0*12,0*5	0.3624	0.4012	0.4226	0.3342	0.3777	0.3618
=			(0.1039)	(0.0986)	(0.0875)	(0.2435)	(0.1046)	(0.1047)
2		0*12,0*5,7*1	0.5877	0.5490	0.5531	0.2213	0.5387	0.5791
			(0.1585)	(0.1206)	(0.1246)	(0.2153)	(0.1105)	(0.1495)
		0*6,0*5,1*7	0.4791	0.4523	0.4599	0.2131	0.4359	0.4745
			(0.0824)	(0.0649)	(0.0670)	(0.2232)	(0.0628)	(0.0780)
		7*1,0*12,0*5	0.3636	0.4024	0.4238	0.3335	0.3790	0.3629
			(0.1029)	(0.0963)	(0.0865)	(0.2317)	(0.1027)	(0.1035)